SKOLEM AND HERBRAND THEOREMS IN FIRST ORDER LOGIC

by

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1. INTRODUCTION.

1.1. The Skolem theorem goes back to SKOLEM 1920. He showed there that if we are willing to use new relationsymbols, then the validity of a formula can be reduced to the validity of a formula of the form

$$(1) \qquad \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m R(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

where R is quantifierfree. If we use new function-symbols instead of relation-symbols we get down to

$$(2) \qquad \exists y_1 \cdots \exists y_m\ S(y_1,\ldots,y_m)$$

where S is quantifierfree. As an example consider

$$(3) \qquad \exists x \forall y \exists z\ P(x,y,z)$$

where P is atomic. To (3) we associate its Skolem transform

$$(4) \qquad \exists x \exists z\ P(x,f(x),z)$$

Here f is a new unary functionsymbol. Then (3) is valid if and only if (4) is.

1.2. Herbrand went a step further in HERBRAND 1929. He got rid of the quantifiers in (2) and (4) by exchanging them with sufficiently large finite disjunctions. Let

$$(5) \qquad \mathcal{D}_n = \{e,f(e),f(f(e)),\ldots\ldots,\ \underbrace{f(f(\cdots\cdots f}_{n}(e)\cdots\cdots))\}$$

Then to formula (3) we associate its n-Herbrand transform

$$(6) \qquad \bigvee_{x\in\mathcal{D}_n} \bigvee_{z\in\mathcal{D}_n} P(x,f(x),z)$$

We then have: A formula is valid if and only if for some n its n-Herbrand transform is valid. This is one formulation of the Herbrand theorem, but it does not capture all in HERBRAND 1929. In section 8.1 below I describe Herbrand results with more details.

<u>1.3.</u> Since 1930 there have been other proofs of Herbrand theorem. The most important are Gentzens verschärfter Hauptsatz (GENTZEN 1934) and the ε-theorems in HILBERT-BERNAYS 1939. Later it was found an essential gap in Herbrands argument (DREBEN, ANDREWS, and AANDERAA 1963). The gap was closed in DREBEN AND DENTON 1966. In NEBRES 1970 part of the natural generalization of Herbrand results to $L_{\omega_1\omega}$ and L_A (A countable admissible) was proved. I describe the results of NEBRES 1970 in section 8.3 below.

<u>1.4.</u> In this paper I shall work with the completeness-proof of BETH 1955, HINTIKKA 1955, SCHÜTTE 1956 and KANGER 1957. Formulated in a calculus of sequents they showed:

a) To any sequent we can construct a "tree of sequents" above it which constitute a systematic attempt to falsify the sequent.
b) If the attempt is successful, we have a falsification of the sequent.
c) If the attempt is not successful, we have a derivation of the sequent. This derivation does not contain cuts.

a,b,c and the soundness of the calculus of sequents, give the completeness of the cut-free reeles. Below I shall develop the theory for the trees considered in a,b,c further. I construct a mapping $\mathcal{S}$ of "trees of sequents" into "trees of sequents such that:

d) $\mathcal{S}$ acts on a one-sequent "tree" by giving the one-sequent "tree" of the Skolem transform of the sequent.

e) $\mathcal{S}$ does not change the tree-structure (i.e. the nodes and branches); but may of course change the sequents at the nodes.

f) If the "tree" T_1 is an extension of the "tree" T_2, then $\mathcal{S}(T_1)$ is an extension of $\mathcal{S}(T_2)$.

g) $\mathcal{S}$ transforms systematic attempts of falsification into systematic attempts of falsification.

h) $\mathcal{S}$ transforms successful attempts of falsification into successful attempts of falsification.

i) $\mathcal{S}$ transforms derivations into derivations.

From the above we get the Skolem theorem. The procedure can be generalized to the infinitary logics $L_{\kappa\omega}$ and L_A (A admissible) without any problems. In fact I will show the above directly for $L_{\kappa\omega}$. The procedure is also a help in pinning down what goes wrong in trying to prove the Skolem theorem in intuitionistic logic and modal logic. I have done this in papers mentioned in the references.

Similar to the transformation $\mathcal{S}$ I construct for each n a transformation $\mathcal{H}_n$ such that:

j) $\mathcal{H}_n$ acts on a one-sequent "tree" by giving the one-sequent "tree" of the n-Herbrand transform of the sequent.

k) $\mathcal{H}_n$ does not change the tree-structure.

l) If the tree T_1 is an extension of T_2, then $\mathcal{H}_n(T_1)$ is an extension of $\mathcal{H}_n(T_2)$.

m) $\mathcal{H}_n$ transforms systematic attempts of falsification into systematic attempts of falsification.

n) $\mathcal{H}_n$ transforms successful attempts of falsification into successful attempts of falsification.

o) If T is a derivation, then from T we can find an n such that $\mathcal{H}_n(T)$ is a derivation.

Using this we get the Herbrand theorem. This carries also over to the infinitary logics with the obvious changes.

In chapter 8 I discuss my results further and compare them with HERBRAND 1929 and NEBRES 1970.

The end of proof is indicated with Ω.

The imprecise notion "tree of sequents" used above will be explicated with classified tree defined in section 3.7.

2. FORMAL SYSTEM FOR $L_{\kappa\omega}$.

2.1. I follow MAEHARA AND TAKEUTI 1961 in describing a formal system for $L_{\kappa\omega}$. Since I am only treating $L_{\kappa\omega}$, not $L_{\kappa\lambda}$, I do not need their complicated eigenvariable condition.- As usual κ is an infinite regular cardinal. (The case that κ is not regular can be treated by working with the next cardinal, κ^+, which is regular.)

A sequence of length κ is called a κ-list, of length $< \kappa$ is called a κ^--list. Sequences will be denoted by symbols like $\{t_\alpha\}_{\alpha<\lambda}$, $\{t_\alpha\}_\alpha$, $\{t_{\alpha,\beta}\}_\alpha$. The last symbol stands for the sequence $t_{0,\beta}$, $t_{1,\beta}$, $t_{2,\beta}$, ...

2.2. The language of $L_{\kappa\omega}$.

a. Parameters. I will use symbols a_λ to stand for parameters. We assume that there are at least κ parameters. This is necessary and sufficient for the proof of the completeness theorem below. Since a derivation in $L_{\kappa\omega}$ may involve more than κ formulaoccurences, it does not seem natural to restrict the language to have only κ parameters.

b. Variables. I will use symbols x_λ to stand for variables.

c. Constantsymbol e. We have one particular constantsymbol e. It will be used in the theory in many places below e.g. in the formulation of the Herbrand theorem. Beside e we get constantsymbols from the 0-place functionsymbols.

d. For each number n, at least κ n-place functionsymbols I will use f,g,h as functionsymbols.
The cardinality condition is necessary and sufficient for the

Skolem theorem.

e. For each number n, n-place predicatesymbols I will use P, Q, R as predicatesymbols.

f. Logical symbols $\neg$, $\bigwedge$, $\forall$

g. I build terms and quasiterms as usual. Terms do not contain variables, quasiterms may contain variables.

h. I build formulas and quasiformulas as usual. A formula is a quasiformula without free variables With $\bigwedge$ I go from a κ^--list of formulas $\{F_\alpha\}_{\alpha<\gamma}$ to a formula $\bigwedge\{F_\alpha\}_{\alpha<\gamma}$. $\bigwedge\{F_\alpha\}_{\alpha<\gamma}$ is also written $\bigwedge_{\alpha<\gamma} F_\alpha$. For $\gamma = 2$ (or $\gamma = 3$) we write $F_0 \wedge F_1$ (or $F_0 \wedge F_1 \wedge F_2$.)

i. A sequent $\Gamma \to \Delta$ is a pair of two (possibly empty) κ^-lists of formulas, Γ, Δ. We use the obvious notations $\Gamma_1, \Gamma_2 \to \Delta$; $\Gamma, F \to \Delta$; $F \to \Delta$; $\to \Delta$; etc. In $\Gamma \to \Delta$ Γ is called the antecedent and Δ the succedent.

<u>2.3.</u> The sequential calculus $L_{\kappa\omega}$ differs from the usual one for $L_{\omega_1\omega}$ in that sequents contain more formulas ($< \kappa$) and we are allowed to change more formulas ($< \kappa$) in each step in the derivations.

I write down the axioms and rules first and comment on them afterwards.

AXIOMS

$\Gamma_1, A, \Gamma_2 \to \Delta_1, A, \Delta_2$ where A is an atomic formula.

STRUCTURAL RULES

Permutation $\dfrac{\Gamma \to \Delta}{\Gamma^{*} \to \Delta^{*}}$ where Γ^{*} is a permutation of Γ and Δ^{*} of Δ.

Contraction $$\frac{\{A_\lambda\}_{\mu<\beta_\lambda\ \lambda<\gamma} \to \{B_\lambda\}_{\nu<\alpha_\lambda\ \lambda<\delta}}{\{A_\lambda\}_{\lambda<\gamma} \to \{B_\lambda\}_{\lambda<\delta}}$$

Trivial rule $$\frac{\Gamma \to \Delta \ \text{-----}\ \Gamma \to \Delta}{\Gamma \to \Delta}$$

LOGICAL RULES

$\neg \to$ $$\frac{\Gamma \to \{A_\lambda\}_\lambda, \Delta}{\Gamma, \{\neg A_\lambda\}_\lambda \to \Delta}$$

$\to \neg$ $$\frac{\Gamma, \{A_\lambda\}_\lambda \to \Delta}{\Gamma \to \{\neg A_\lambda\}_\lambda, \Delta}$$

$M \to$ $$\frac{\Gamma, \{A_{\lambda,\mu_{\alpha,\lambda}}\}_{\alpha<\delta_\lambda\ \lambda<\gamma} \to \Delta}{\Gamma, \{\underset{\mu<\beta_\lambda}{M} A_{\lambda,\mu}\}_{\lambda<\gamma} \to \Delta}$$ where $\mu_{\alpha,\lambda}<\beta_\lambda$ for $\lambda<\gamma$

$\to M$ $$\frac{\text{--}\Gamma \to \{A_{\lambda,\mu_\lambda}\}_{\lambda<\gamma}, \Delta\ \text{--}}{\Gamma \to \{\underset{\mu<\beta_\lambda}{M} A_{\lambda,\mu}\}_{\lambda<\gamma}, \Delta}$$ for all sequences $\{\mu_\lambda\}_{\lambda<\gamma}$ with $\mu_\lambda<\beta_\lambda$ for $\lambda<\gamma$

$\forall \rightarrow$ $$\frac{\Gamma, \{A_\lambda t_{\lambda,\mu}\}_{\mu<\beta_\lambda, \lambda<\gamma} \rightarrow \Delta}{\Gamma, \{\forall x_\lambda A_\lambda x_\lambda\}_{\lambda<\gamma} \rightarrow \Delta}$$

$\rightarrow \forall$ $$\frac{\Gamma \rightarrow \{A_\lambda a_\lambda\}_{\lambda<\gamma}, \Delta}{\Gamma \rightarrow \{\forall x_\lambda A_\lambda x_\lambda\}_{\lambda<\gamma}, \Delta}$$, where the parameters are all distinct and do not occur in $\Gamma \rightarrow \{\forall x_\lambda A_\lambda x_\lambda\}_{\lambda<\delta}, \Delta$

2.4. With all the indices the rules are rather difficult to digest. I shall try to show with examples and comments that they are all straightforward.

2.4.1. The axioms and the permutationrule is as usual for a sequential calculus.

2.4.2. Contraction. As a special case assume $\gamma = 2$, $\beta_0 = 2$, $\beta_1 = 3$, $\delta = 2$, $\alpha_0 = 2$, $\alpha_1 = 0$. Then

$$\frac{A_0, A_0, A_1, A_1, A_1 \rightarrow B_0, B_0}{A_0, A_1 \rightarrow B_0, B_1}$$

Note that the usual thinning rule is a special case of our contractionrule. (Above we introduce B_1 by some sort of thinning.)

2.4.3. Trivial rule. This rule is certainly trivial

2.4.4. $\rightarrow \neg$ and $\neg \rightarrow$. As an instance of $\neg \rightarrow$ we have:

$$\frac{\Gamma, \rightarrow A_0, A_1, A_2, \Delta}{\Gamma, \neg A_0, \neg A_1, \neg A_2 \rightarrow \Delta}$$

2.4.5. $M \rightarrow$. An instance is :

$$\frac{\Gamma, A_{0,2}, A_{0,2}, A_{1,0}, A_{1,1} \rightarrow \Delta}{\Gamma, A_{0,0} \wedge A_{0,1} \wedge A_{0,2}, A_{1,0} \wedge A_{1,1} \rightarrow \Delta}$$

2.4.6. $\rightarrow M$. An instance is :

$$\frac{\Gamma \rightarrow A_{0,0}, A_{1,0} \quad \Gamma \rightarrow A_{0,0}, A_{1,1} \quad \Gamma \rightarrow A_{0,1}, A_{1,0} \quad \Gamma \rightarrow A_{0,1}, A_{1,1}}{\Gamma \rightarrow A_{0,0} \wedge A_{0,1},\ A_{1,0} \wedge A_{1,1}}$$

In the non-trivial case that all $\beta_\lambda \geq 2$ we will have more than $|2^\gamma|$ premisses. So a tree of sequents can become very large. It is therefore important to single out those sequential calculi where we can work with only finite sequents (and hence also finite γ) without destroying completeness. We will come back to this problem in section 4.7.

2.4.7. $\forall \rightarrow$. An instance is :

$$\frac{\Gamma, A_0 t_{0,1}, A_1 t_{1,2}, A_1 t_{1,0} \rightarrow \Delta}{\Gamma, \forall x_4 A_0 x_4,\ \forall x_3 A_1 x_3 \rightarrow \Delta}$$

2.4.8. $\rightarrow \forall$. An instance is :

$$\frac{\rightarrow A_0 a_3,\ A_1 a_1}{\rightarrow \forall x_1 A_0 x_1,\ \forall x_4 A_1 x_4}$$

<u>2.5.</u> Derivations in $L_{\kappa\omega}$ are defined as usual and we write ' $\vdash_{L_{\kappa\omega}} \Gamma \rightarrow \Delta$ ' for ' $\Gamma \rightarrow \Delta$ is derivable in $L_{\kappa\omega}$.' To develop the theory sketched in section 1.4 we must be more precise. Therefore we introduce ' trees of sequents ' and ' classical trees ' in the next chapter.

3. TREES OF SEQUENTS.

3.1. Our trees have exactly one downmost node and are spreading upwards from this node. Below any node there are only finitely many nodes.

A tree of sequents is a tree with sequents at the nodes and for each node ν if it has immediate successions $\nu_1, \nu_2, \ldots$, then the sequent at ν and the sequents at $\nu_1, \nu_2, \ldots$ are related as conclusion and premisses of one of the rules of $L_{\kappa\omega}$. A tree of sequents is then given by :

a) its treestructure ;

b) for each node ν, the sequent at ν; and

c) for each node ν which is not topmost, the instance of the rule which has the sequent at ν as its conclusion.

Note that we do neither assume that the trees are well-founded nor anything about occurrences of axioms.

DEFINITION

The tree of sequents T_1 is an extension of the tree of sequents T_2 if

i) every node in T_2 is also in T_1; and

ii) for every node ν in T_2 and every node μ below ν in T_1, μ is also in T_2.

3.2. Now consider the following tree of sequents :

$$\dfrac{\dfrac{A \overset{3}{\rightarrow} A, A}{A \wedge B \overset{2}{\rightarrow} A, A} \qquad \dfrac{B \overset{5}{\rightarrow} B, A}{A \wedge B \overset{4}{\rightarrow} B, A}}{A \wedge B \overset{1}{\rightarrow} A \wedge B, A}$$

I have indicated the nodes with the numbers 1-5 above the arrows. The rules with the sequents at 1,2,4 as conclusions are $\rightarrow M$, $M \rightarrow$, $M \rightarrow$. In the tree there are 3 formulas: A,B , $A \wedge B$, while there are 15 formulaoccurrences - 3 occurrences at each of the 5 nodes. Now we would like to give an equivalence relation between formulaoccurrences having similar functions in the tree of sequents. One should certainly relate all 5 occurrences of A last in the succedents, but it does not seem natural to relate all 3 occurrences of A in node 3. The 3 A's have different roles in the tree of sequents. Only the last A in node 3 is natural to relate to the A in node 1.- I want to relate two formulaoccurrences if they are occurrences of the same formula in the same way. This equivalence relation will be made precise below with ' two formulaoccurrences being in the same fibre '. The tree of sequents above will have 7 fibres - 3 with occurrences of A, 2 with occurrences of B, and 2 with occurrences of $A \wedge B$.- When we use quantifiers the notion of fibre becomes slightly more complicated.

In section 3.6 comes the definition of ' being in the same fibre '. First we must make clear how to trace formulaoccurrences in a tree of sequents.

3.3. Tracing in the rules of $L_{\kappa\omega}$.

Consider the rules of $L_{\kappa\omega}$ as given in section 2.3. The formulaoccurrences in Γ and Δ are called sideformulas while the A's and B's are called mainformulas.

To each occurrence of a sideformula F in a premiss, there is in a natural way one occurrence of a sideformula F^* in the conclusion where F and F^* are occurrences of the same formula. We say that :

a) F immediately succeds F^* as formula ;

b) F^* immediately precedes F as formula.

To each occurrence of a part P of F there is a corresponding occurrence P^* of a part of F^*. P and P^* are occurrences of the same formulapart. We say

c) P immediately succeds P^* as formulapart;

d) P^* immadiately precedes P as formulapart.

With the mainformulas it is slightly more complicated. Consider for example a mainformula $A_1 t_{1,2}$ of the premiss of an instance of $\forall \rightarrow$. To it corresponds exactly one occurrence $\forall x_{\delta_1} A_1 x_{\delta_1}$ in the conclusion. We say that :

e) the occurrence of $A_1 t_{1,2}$ immediately succeds the occurrence of $\forall x_{\delta_1} A_1 x_{\delta_1}$ as formula ;

f) The occurrence of $\forall x_{\delta_1} A_1 x_{\delta_1}$ immediately precedes the occurrence of $A_1 t_{1.2}$ as formula.

To each part P of $A_1 t_{1,2}$ there corresponds a part P^* of $\forall x_{\delta_1} A_1 x_{\delta_1}$ such that P and P^* are occurrences of the same parts except possibly for some occurrences of $t_{1,2}$ instead of x_{δ_1}. For P and P^* we again have c and d. The mainformulas in other rules are treated similarly.

We have now explained

' immediately succeds as formula '

' immediately precedes as formula '

' immediately succeds as formulapart '

' immediately precedes as formulapart ' .

3.4. Tracing in trees of sequents.

'Immediately succeds/precedes as formula/formulapart' is carried over to trees of sequents without any changes. We take '<u>succeds/precedes as formula formulapart</u>' as the transitive and reflexive closure.

Two occurrences of formulas are <u>in the same strand of formulas</u> if there is a formulaoccurrence which precedes both as formula.

Two occurrences of formulaparts are <u>in the same strand of formulaparts</u> if there is a formulapartoccurrence which precedes both as formulapart.

3.5. Keeping track of quantifiers.

To each mainformula F in the premiss of a quantifierrule (i.e. $\forall \rightarrow$ or $\rightarrow \forall$) we associate the pair $\langle x,t \rangle$ where t is the term we have inserted for the variable x to get F.

Now to each formulaoccurrence G in a tree of sequents we define :

The <u>general (restricted) analysis</u> G is the sequence of pairs associated to formulaoccurrences H such that

a) H precedes G as formula ; and

b) H is a mainformula in the premiss of a rule $\rightarrow \forall$ ($\forall \rightarrow$).

The <u>analysis</u> of G is the pair of the general and the restricted analysis of G.

3.6. Fibres.

Given a tree of sequents. Two formulaoccurrences F_1 and F_2 are in the same fibre if

a) F_1 and F_2 are in the same strand of formulaparts; and

b) F_1 and F_2 have the same analysis.

This is clearly an equivalence relation. The equivalence classes are called fibres.

Now going back to the example of 3.2 we can immediately count the 7 fibres. Since we do not have quantifiers in the example, we do not need condition 6 in the definition of fibre. Below we give an example with 11 fibres :

$$\dfrac{\dfrac{\dfrac{A,B,\forall yR(fa,y),R(fa,fe) \to C,A}{A,B,\forall yR(fa,y),\forall yR(fa,y) \to C,A}}{A,B,\forall yR(fa,y) \to C,A} \qquad \dfrac{\dfrac{A,B,\forall yR(fa,y),R(fa,fe) \to D,A}{A,B,\forall yR(fa,y),\forall yR(fa,y) \to D,A}}{A,B,\forall yR(fa,y) \to D,A}}{\dfrac{\dfrac{A,B,\forall yR(fa,y) \to C\wedge D,A}{A,B \to \neg\forall yR(fa,y)\ ,\ C\wedge D,A}}{\dfrac{A\wedge B \to \neg\forall yR(fa,y)\ ,\ C\wedge D,A}{A\wedge B \to \forall x\neg\forall yR(fx,y),C\wedge D,A}}}$$

The following easy lemma is often used without explicitly mentioning it :

LEMMA

If two formulaoccurrences are in the same fibre, then they are occurrences of the same formula and they are both in succedent or they are both antecedent.

For later use we note that the strands of formulaparts can be identified with the subformulas in the bottomsequents. Hence the fibres can be identified with ordered pairs $<F_1<<x_1,t_1>,\cdots,<x_n,t_n>>$ where F is a subformula of the bottomsequent and $<<x_1,t_1>,\cdots,<x_n,t_n>>$ is a finite sequent of pairs $<x_i,t_i>$ with x_i a variable in the bottomsequent and t_i a term.

3.7. Classical trees.

For two nodes μ, ν in a tree of sequents we say that μ is at a lower level than ν if the height of μ over the bottomnode is $<$ than the height of ν over the bottomnode. They are of the same level if they have the same height over the bottomnode.

Consider the rules $\rightarrow \forall$ and $\forall \rightarrow$ described in 2.3. The terms introduced by $\rightarrow \forall (\forall \rightarrow)$ are $\{a_\lambda\}_{\lambda<\gamma}$ $(\{t_{\lambda,\mu}\}_{\lambda<\beta_\lambda \beta<\gamma})$
The term t is introduced by $\rightarrow \forall (\forall \rightarrow)$ at node μ in the tree of sequents T if μ is the premiss of the rule $\rightarrow \forall (\forall \rightarrow)$ and t is among the terms introduced by the rule.

A classical tree T is a tree of sequents such that :

a) any term introduced by $\forall \rightarrow$ at node ν in T is built up from the constant e, symbols from the bottomsequent of T, and parameters introduced by $\rightarrow \forall$ in T at lower level than ν; and

b) any two parameters introduced by $\rightarrow \forall$ are equal only if in the nodes where they are introduced they occur in formulaoccurrences in the same fibres.

A classical tree over $\Gamma \rightarrow \Delta$ is a classical tree with $\Gamma \rightarrow \Delta$ at the bottomnode.

The classical trees will be our working material. The theory hinted at in 1.4 is developed for classical trees.

4. COMPLETENESS OF $L_{\kappa\omega}$.

4.1. The completeness proof follows the key ideas of BETH 1955, HINTIKKA 1955, SCHÜTTE 1956 and KANGER 1957 described in 1.4 above.

4.2. A branch in a tree is a path going from the downmost node and as far as possible upwards.

A node in a tree of sequents is a secured node if there is an axiom at it.

A branch in a tree of sequents is a secured branch if it contains a secured node.

A tree of sequents is a secured tree of sequents if all its branches are secured.

DEVIABILITY THEOREM

For any sequent $\Gamma \rightarrow \Delta$: $\vdash \Gamma \rightarrow \Delta$ if there is a secured classical tree over $\Gamma \rightarrow \Delta$.

The proof is obvious. For the converse we must prove : "If there is a secured tree of sequents over $\Gamma \rightarrow \Delta$, then there is a secured classical tree over $\Gamma \rightarrow \Delta$." This follows from the completeness proof below, but can also be given directly assuming the language has enough parameters.

THEOREM

If T_1 is a secured tree of sequents over $\Gamma \rightarrow \Delta$, then there is a secured classical tree T_2 over $\Gamma \rightarrow \Delta$.

Proof: Let T_1 be a secured tree of sequents over $\Gamma \rightarrow \Delta$. For each parameter a in T_1 we introduce the following binary relations between occurrences of a in T_1 :

$\rho_a(o_1, o_2) \Leftrightarrow$ the occurrences o_1 and o_2 of a are either in the same node, or o_1 is in a node immediately above o_2.

σ_a is the transitive, symmetric, reflexive closure of ρ_a. Let $[o]_a$ be the equvalence class belonging to o under σ_a.

Let $I = \{[o]_a \mid o$ is an occurrence of parameter $a\}$

Let $h : I \to$ Parameters be 1-1 such that for each occurrende o of a in the bottomsequent $h([o]_a) = a$.

We then get $I_1{}^*$ from T_1 by mapping each occurrende o of a in T_1 into $h([o]_a)$.

It is straightforward that $T_1{}^*$ is a secured tree of sequents such that parameters introduced by $\to \forall$ at different nodes in $T_1{}^*$ are different.

We now call a redundant term a term in $T_1{}^*$ which contains a symbols which is neither

1) in $\Gamma \to \Delta$;
2) the constant symbol e ; nor
3) among the parameters introduced by $\to \forall$.

We then get T_2 from $T_1{}^*$ by mapping redundant terms into the constant symbol e.

The proof that T_2 is a secured classical tree over $\Gamma \to \Delta$ is an easy exercise in the notions introduced so far. Ω

4.3. Analyzing trees.

Now we want to explicate "systematic attempt of falsification". To do this we use the special types of classical trees defined below -

analyzing trees. The analyzing trees will only in trivial cases be well-founded. We say that a formulaoccurrence occurs as a successor to a fibre if it occurs as a successor to a formula-occurrence in the fibre.

ANALTZING BRANCH

A branch β in a classical tree T is an analyzing branch when :

i) if (the formula) $\neg F$ occurs in β, then F occurs in β as successor to the fibre of $\neg F$;

ii) if MF_i occurs in an antecedent in β, then each F_i occurs in β as a successor to the fibre of MF_i;

iii) if MF_i occurs in a succedent in β, then one of the F_i occurs in β as a successor to the fibre of MF_i;

iv) if $\forall xFx$ occurs in an antecedent in β, then for each term t built up from symbols in T, Ft occurs in β as a successor to the fibre of $\forall xFx$;

v) if $\forall xFx$ occurs in a succedent in β, then for some term t Ft occurs in β as a successor to the fibre of $\forall xFx$; and

vi) if an atomic formula A occurs at node ν in β, then for each node μ in β higher up than ν, A occurs at μ as a successor to A.

ANALYZING TREE

A classical tree is an analyzing tree if every branch in it is analyzing.

The correspondence with the notions in 1.4 a,b,c is as follows:

i) "Systematic attempt of falsification of $\Gamma \rightarrow \Delta$" is "analyzing tree over $\Gamma \rightarrow \Delta$".

ii) "Successful attempt" is "analyzing tree with not-secured branch".

iii) "Not successful attempt" is "secured analyzing tree".

In the next section we will prove that to any sequent we can find an analyzing tree over it. That gives 1.4 a. 1.4 c is given by the derivability theorem in the previous section. Here we give 1.4 b :

FALSIFIABILITY THEOREM

Let T be a classical tree over $\Gamma \to \Delta$. If we have a not-secured analyzing branch β in T, then there is a falsifying model of $\Gamma \to \Delta$ (i.e. a model where all the formulas in Γ are true and all those in Δ false).

Proof: Assume we have such a β.

The model is constructed as follows :

The domain is the set of terms built up from symbols in T. An atomic formula is true if and only if it occurs in an antecedent in β.

Then it is easily proved by induction over the length of formulas that in the model any formula occurring in an antecedent in β is true and any formula occurring in a seccedent in β is false. Ω

<u>4.4.</u> Construction of analyzing trees.

We have given a sequent $\Gamma \to \Delta$ (in $L_{\kappa\omega}$) and shall construct an analyzing tree T over it. The construction goes by $\omega + 1$ stages. After each finite stage N we have a well-founded classical tree T_N. T_{N+1} is an extension of T_N. $T = \bigcup_N T_N$.

We write down the stages in shorthand below and interprete it afterwards.

STAGE 0 $\quad T_o$ consists of the sequent $\Gamma \to \Delta$ alone.

STAGE 6N+1 :

$$\begin{array}{c}
\Gamma_2 \to \{A_\alpha\}_\alpha, \Delta_1 \\ \hline
\Gamma_2, \{\neg A_\alpha\}_\alpha \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1
\end{array}$$

STAGE 6N+2 :

$$\begin{array}{c}
\Gamma_1, \{A_\alpha\}_\alpha \to \Delta_2 \\ \hline
\Gamma_1 \to \{\neg A_\alpha\}_\alpha, \Delta_2 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1
\end{array}$$

STAGE 6N+3 :

$$\begin{array}{c}
\Gamma_2, \{A_{\alpha,\beta}\}_{\beta<\gamma_\alpha, \alpha} \to \Delta_1 \\ \hline
\Gamma_2, \{\bigwedge_{\beta<\gamma_\alpha} A_{\alpha,\beta}\}_\alpha \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1 \\ \hline
\Gamma_1 \to \Delta_1
\end{array}$$

STAGE 6N+4 :

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \to \{A_{\alpha,\mu_\alpha}\}_\alpha, \Delta_2 \quad \text{for all} \quad \{\mu_\alpha\}_\alpha \quad \text{with} \quad \mu_\alpha < \gamma_\alpha}{\Gamma_1 \to \{\bigwedge_{\beta<\gamma_\alpha} A_{\alpha,\beta}\}_\alpha, \Delta_2}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \to \Delta_1}$$

STAGE 6N+5 :

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \ \{A_\alpha t_{\beta,\alpha}\}_{\alpha,\beta} \to \Delta_1}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \ \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha, \forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_1 \to \Delta_1}$$

where for each α $\{t_{\beta,\alpha}\}_\beta$ is a κ^--sequence without repetition of all the terms of length $< 6N+5$ built up from the symbols in T_{6N+4} and the constant e and such that no $A_\alpha t_{\beta,\alpha}$ occurs as successor to the fibre of $\forall x_\alpha A_\alpha x_\alpha$ in T_{6N+4}.

STAGE 6N+6 :

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \to \{A_\alpha a_{F_\alpha}\}_\alpha, \Delta_2}{\Gamma_1 \to \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \Delta_2}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \to \Delta_1}$$

where F_α is the fibre of $\forall x_\alpha A_\alpha x_\alpha$. We will prove below that

1) The F_α above any topmost node in T_{6N+5} are distinct.

2) In all the topmost nodes in T_{6N+5} there are $< \kappa$ F_α's.

3) None of the a_{F_α} occur in T_{6N+5}.

This concludes the "shorthand description" of the construction of T.

INTERPRETATION

The shorthand above for stage 6N+1 is interpreted as follows :

In stage 6N+1 we start with the tree of sequents T_{6N}. For each of the topmost nodes ν in T_{6N} we do : Say that the sequent $\Gamma_1 \rightarrow \Delta_1$ is at ν.

1) If there are no $\neg$ - formulas in Γ_1, then tack $\Gamma_1 \rightarrow \Delta_1$ four times above ν to get :

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma_1 \rightarrow \Delta_1}{\Gamma_1 \rightarrow \Delta_1}}{\Gamma_1 \rightarrow \Delta_1}}{\Gamma_1 \rightarrow \Delta_1}}{\Gamma_1 \not\rightarrow \Delta_1}$$

2) If $\{\neg A_\alpha\}_\alpha$ are the $\neg$ - formulas in Γ_1, then by permutation of Γ_1 we get $\Gamma_2, \{\neg A_\alpha\}_\alpha$. Now above node ν we tack on :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_2 \to \{A_\alpha\}_\alpha, \Delta_1}{\Gamma_2\{\neg A_\alpha\}_\alpha \to \Delta_1}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \to \Delta_1}}{\Gamma_1 \not\to \Delta_1}}{}$$

Having done this for each topmost node in T_{6N} we get T_{6N+1}.

The other finite stages are interpreted in the same way with only the obvious changes. If it is not possible to do one stage, we stop.

The regularity of κ in $L_{\kappa\omega}$ is assumed in this construction in stages $6N+3$, $6N+5$. We need the fact that a κ^{-}-sequence of $\bar{\kappa}$-sequences is a $\bar{\kappa}$-sequence, to prove that we get sequents (of $L_{\kappa\omega}$) at the nodes in T_{M+1} provided we have it in T_M.

LEMMA 1.

For finite N: T_N is of height $4N+1$ and hence is well-founded.

LEMMA 2.

For finite N: If T_N contains less than κ symbols, then T_N contains less than κ fibres.

Proof: Assume T_N contains less than κ symbols. In T_N we only introduce terms of length $< N$. Each branch in T_N is of length $4N+1$, and hence for any formulaoccurrence in T_N there are

$\leq 4N$ uses of $\forall\rightarrow$ or $\rightarrow\forall$ preceding it. The fibres in T_N can be identified with ordered pairs $\langle s,F\rangle$ where s is a sequence of length $\leq 4N$ of terms of length $< N$ built up from $< \kappa$ symbols, and F is a subformula of $\Gamma \rightarrow \Delta$.

There are $< \kappa$ such sequences s and $< \kappa$ such subformulas F (use regularity of κ).

Hence there are $< \kappa$ fibres in T_N. Ω

LEMMA 3.

For any branch β in T and any fibre F which occurs in β, F has a unique downmost (with respect to predessor) formula-occurrence.

Proof:

By induction.

The lemma is true for fibres which contain formulaoccurrences from the bottomsequent.

Assume the lemma true for fibre F.

We shall show that it is true for all fibres G of formala-occurrences which are immediate successors of F.

We divide up into cases depending on the outermost symbol in F and whether F occurs in antecedent or succedent.

We go through the case that F is the fibre of $\forall xGx$ in antecedent of β and G is Gt. By assumption F has a unique downmost formulaoccurrence in β, say at node μ. μ occurs first in T_N in the construction of T.

Let $6M+5$ be the least number $> N$ such that

i) the symbols in t with possible exception of e occur in T_{6M+4}; and

ii) the length of $t < 6M+5$.

At the topmost node along β in each T_κ for $\kappa \geq N$ there occurs an $\forall xGx$ in fibre F.

From the construction we see that at stage $6M+5$ we introduce the first formulaoccurrence of G in β and at no later stages do we introduce new formulaoccurrences of G in β.

It follows that also G has a unique downmost formulaoccurrence in β.

Induction completed. Ω

From lemma 3 we get:

LEMMA 4.

For each finite N, each fibre F, each topmost node μ in T_N there is at most one formulaoccurrence both in F and μ.

The introduction of fibres in T is not only well behaved along the branches, but also across them.

LEMMA 5.

For any fibre F, the downmost formulaoccurrences of F in branches of T are all at nodes of the same level.

Proof: By similar induction as in lemma 3.

The lemma is true for fibres of formulaoccurrences in the bottomsequent.

Assume lemma true for fibre F.

Let G be a fibre of a formulaoccurrence which is an immediate successor of F.

By inductionassumption all downmost occurrences of F are at node of say height N.

We now divide up into cases depending on the outermost symbol

of F and whether F occurs in antecedent or succedent.
Say that F is the fibre of $\neg A$ in antecedent and G is the fibre of A.

Let $6M+1$ be the least number with $4\cdot(6M+1)+1 > N$. Then all downmost occurrences of G are at nodes of height $4\cdot(6M+1)+1$.

The other cases are similar. Ω

THEOREM

The construction of T can be carried through and for each finite N, T_N has $< \kappa$ symbols.

Proof: By induction over the stages.

Obvious for STAGE 0.

Assume it proved for STAGE N. I shall prove it for STAGE N+1.

If $N+1 = 6M+1, 6M+2, 6M+3, 6M+4$, then the proof is obvious.

If $N+1 = 6M+5$, then we use that T_N has less than κ symbols to get the κ^{-}-sequences of terms.

If $N+1 = 6M+6$, then we use the lemmas. For each fibre F, let a_F be a parameter. To each $\forall x A x$ in the topmost node in succedent in T_N we assign parameter a_F where F is the fibre of $\forall x A x$. This assignment of parameters is easily seen by the lemmas to give the wanted properties 1 - 3 of STAGE 6M+6.

STAGE ω is obvious. Ω

ANAYLIZING THEOREM

To any sequent $\Gamma \rightarrow \Delta$ (in $L_{\kappa\omega}$), we can find an analyzing classical tree over it.

Proof: First we use the construction above to get the classical tree T over $\Gamma \rightarrow \Delta$.

Then we check that T is analyzing.

Let β be a branch in T. We shall prove that β is analyzing. As example we verify clause iv) in the definition of analyzing branch :

Assume that the formula $\forall xFx$ occurs in an antecedent at node ν in β and that t is a term built up from symbols in T and the constant symbol e. Let μ be the lowest node in β where there occurs an $\forall xFx$ in the same fibre as the $\forall xFx$ at ν, and N_0 be the least number such μ is in T_{N_0}

Say that T is of length N_1 and N_2 is the least number such that the symbols of t occurs in T_{N_2}. Let M be the least number with $6M+5 > \max(N_0, N_1, N_2)$. Then at stage $6M+5$ we introduce in β an occurrence of Ft as a successor to the fibre of $\forall xFx$.

Ω

It is the analyzing theorem that we need for the completeness theorem. The construction of T above gives more.

STRONG CLASSICAL TREE.

A classical tree S is strong if the parameters introduced by $\to \forall$ are equal if and only if they are introduced by formula-occurrences in the same fibre.

We obviously have:

LEMMA T is strong

and

STRONG ANALYZING THEOREM.

For any sequent $\Gamma \to \Delta$ we can construct a strong analyzing classical tree over it.

This theorem is a key to the Skolem theorem. More about this in the next chapter. In modal logic and intuitionistic logic we can make analogues to the analyzing theorem, but then the strong analyzing theorem fails. (See my papers on intuitionistic and modal logic mentioned in the references.)

4.5. Completeness of $L_{\kappa\omega}$.

COMPLETENESS THEOREM

For any sequent $\Gamma \rightarrow \Delta$: $\vdash \Gamma \rightarrow \Delta$ iff $\Gamma \rightarrow \Delta$ cannot be falsified.

Proof:

1) Assume $\vdash \Gamma \rightarrow \Delta$. By induction over the derivation we prove that it cannot be falsified.

2) Assume not $\vdash \Gamma \rightarrow \Delta$.
By the analyzing theorem there is an analyzing classical tree T over $\Gamma \rightarrow \Delta$.
T is not secured since else we would have $\vdash \Gamma \rightarrow \Delta$ by the derivability theorem.
There must be an analyzing not-secured branch β in T.
By the falsiability theorem $\Gamma \rightarrow \Delta$ can be falsified. Ω

We also note

COROLLARY

For any sequent $\Gamma \rightarrow \Delta$ exactly one of the following must be true:

1) There is a secured tree over $\Gamma \rightarrow \Delta$.

2) There is a not-secured analyzing tree over $\Gamma \rightarrow \Delta$.

Proof: By the analyzing theorem at least one of 1 or 2 must be true. Assume both 1 and 2 true.

By 1 and the derivability theorem $\vdash \Gamma \to \Delta$

By 2 and the falsifiability theorem not $\vdash \Gamma \to \Delta$

Hence at most one of 1 or 2 is true. Ω

It is an interesting fact that both derivability and not derivability are equivalent to statements about existence of classical trees with certain properties. Using this we concentrate on transformations of classical trees preserving those properties. Both the Skolem theorem and the Herbrand theorem are done by developing theories for transformations of trees.

4.6. Finitary completeness.

If we use large enough branchings in the trivial rule, we can have as many nodes in the derivations of $L_{\kappa\omega}$ as we want to. But even if we restrict the trivial rule to have $< \kappa$ premisses, we can have up to $\mathrm{Inacc}(\kappa)$ nodes, where $\mathrm{Inacc}(\kappa)$ is the first inaccessible $\geq \kappa$.

With such huge trees it is impossible to code them in an elementary way, and they will not be definable in any weak set theory. The huge trees can of course be avoided in $L_{\omega\omega}$ since $\mathrm{Inacc}(\omega) = \omega$. The problems come in $L_{\omega_1\omega}$. It turns out that there we can avoid the problems with huge trees by restricting ourselves to trees where all sequents are finite.

DEFINITION

A finitary classical tree of $L_{\kappa\omega}$ is a classical tree where in the trivial rule we have $< \kappa$ premisses and where all the sequents are finite.

LEMMA

A finitary classical tree of $L_{\kappa\omega}$ has $< \kappa \times \omega_1$ nodes.

Proof: By induction over N we prove that a finitary classical tree of $L_{\kappa\omega}$ has $\leq \kappa$ nodes of height N over the bottomsequent.

The inductionstart is obvious.

The inductionstep follows easily as soon as we note that all rules in a finitary classical tree have $< \kappa$ premisses.

The induction is completed and the lemma follows. Ω

The problem now is whether we can get finitary completeness for $L_{\kappa\omega}$ - that is : a finite sequent is finitary derivable if and only if it cannot be falsified. There are many ways to see that this cannot be done in general. For example we have the following argument, a - g below, from TAIT 1968 :

a) Let P_N^0, $N \in NAT$ be o-place relationsymbols and for each N write P_N^1 for $\neg P_N^0$. Here NAT is the set of natural numbers.

b) Let 2^{NAT} be the uncountable set of all functions $NAT \to \{0,1\}$.

c) We obviously have that $\to \neg \underset{f \in 2^{NAT}}{M} \neg \underset{N \in NAT}{M} P_N^{f(N)}$ cannot be falsified, and hence by the completeness theorem

$$\vdash \to \neg \underset{f \in 2^{NAT}}{M} \neg \underset{N \in NAT}{M} P_N^{f(N)} .$$

d) Let us now assume that $\to \neg \underset{f \in 2^{NAT}}{M} \neg \underset{N \in NAT}{M} P_N^{f(N)}$ can also be finitarily derived. We can assume that the finitary derivation D is a well-founded tree where we do not use the trivial rule.

e) In the finitary derivation D the only places where we have branchings are uses of $\to \bigwedge$ of the form :

$$\frac{\Gamma \to \{P_{N_\alpha}^{f_\alpha(N_\alpha)}\}_{\alpha<\beta}, \Delta \quad \text{for all } \{N_\alpha\}_{\alpha<\beta} \text{ with } N_\alpha \in NAT}{\Gamma \to \{\bigwedge_{N \in NAT} P_N^{f_\alpha(N)}\}_{\alpha<\beta}, \Delta}$$

Since D is a finitary derivation, β is finite. Therefore we have only a countable branching here, and in D all branchings are countable. We conclude that D contains only countable many nodes.

f) Since there are only countably many nodes in D, there are also only countably many uses of $\bigwedge \to$ in D. Each of the uses are of the form :

$$\frac{\Gamma_1\{\neg \bigwedge_{N \in NAT} \neg P_N^{f_{\alpha,\beta}(N)}\}_{\alpha<\gamma_\beta, \beta<\delta} \to \Delta}{\Gamma_1\{\bigwedge_{f \in 2^{NAT}} \neg \bigwedge_{N \in NAT} P_N^{f(N)}\}_{\beta<\delta} \to \Delta}$$

Now in all the nodes we have only countable many functions $f_{\alpha,\beta} \in 2^{NAT}$ which we use in D. Hence there is a countable subset $F \subseteq 2^{NAT}$ such that all the functions used in $\bigwedge \to$ D are in F. From C we get a derivation of $\to \neg \bigwedge_{f \in F} \neg \bigwedge_{N \in NAT} P_N^{f(N)}$. But this sequent isfalsifiable since F is countable. This contradicts the completeness theorem.

g) We conclude that $\to \neg \bigwedge_{f \in 2^{NAT}} \neg \bigwedge_{N \in NAT} P_N^{f(N)}$ is derivable but not finitary derivable. The finitary completeness does not hold in general.

Another argument is the following :

a) For finitary derivations in $L_{\kappa\omega}$ we can prove Craigs interpolation theorem. See for example FEFERMAN 1968.

b) Now the semantical formulation of Craigs interpolation theorem is false for $L_{\kappa\omega}$, $\kappa > \omega_1$. An argument for this is in MALITZ 1965.

c) We conclude that finitary completeness does not hold for $L_{\kappa\omega}$ $\kappa > \omega_1$.

The argument for finitary completeness of $L_{\omega_1\omega}$ is well known (LOPEZ-ESCOBAR 1965) but is worth repeating. The key is to construct finitary analyzing trees for finite sequencts in $L_{\omega_1\omega}$.

a) We start with the finite sequent $\Gamma \to \Delta$ in $L_{\omega_1\omega}$

b) Since $\Gamma \to \Delta$ is in $L_{\omega_1\omega}$ we can assume that all conjunctions are of the form $\bigwedge_{n<\omega} A_n$.

c) In the tree T^* that we want to construct over $\Gamma \to \Delta$ all branches will be countable and we can assume that all nodes in T^* are ordered in an ω-sequence.

d) STAGE 6N+3 is exchanged with

*-STAGE 6N+3

$$\begin{array}{r}
\Gamma_2, \{\bigwedge_{n<\omega} A_{n,\alpha}\}_\alpha, \{A_{n,\alpha}\}_{n<N,\alpha} \quad \to \Delta_1 \\
\hline
\Gamma_2, \{\bigwedge_{n<\omega} A_{n,\alpha}\}_\alpha, \{\bigwedge_{n<\omega} A_{n,\alpha}\}_\alpha \quad \to \Delta_1 \\
\hline
\Gamma_2, \{\bigwedge_{n<\omega} A_{n,\omega}, \bigwedge_{n<\omega} A_{n,\omega}\}_\omega \quad \to \Delta_1 \\
\hline
\Gamma_2, \{\bigwedge_{n<\omega} A_{n,\alpha}\}_\alpha \quad \to \Delta_1 \\
\hline
\Gamma_1 \quad \to \Delta
\end{array}$$

e) STAGE 6N+5 is exchanged with

*-STAGE 6N+5

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \{A_\alpha t_{\alpha,\beta}\}_{\alpha,\beta} \to \Delta_1}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha, \forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_2, \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}}{\Gamma_1 \to \Delta_1}$$

where $\{t_{\beta,\alpha}\}_\beta$ is a finite sequence of the term $t_{\beta,\alpha}$ such that

i) length of $t_{\beta,\alpha} < 6N+5$;

ii) $t_{\beta,\alpha}$ is built up from symbols in the first 6N+5 nodes (in the ω-sequence of nodes in T^*) which are in T^*_{6N+4} and the constant e ; and

iii) no $A_\alpha t_{\beta,\alpha}$ occurs in T^*_{6N+4} as successor to the fibre of $\forall x_\alpha A_\alpha x_\alpha$.

f) The other stages are as before.

Using this construction we clearly get

FINITARY ANALYZING THEOREM FOR $L_{\omega_1\omega}$.

For any finite sequent in $L_{\omega_1\omega}$ we can construct a finitary analyzing classical tree over it.

and

FINITARY COMPLETENESS OF $L_{\omega_1 \omega}$.

For any finite sequent $\Gamma \rightarrow \Delta$ in $L_{\omega, \omega}$ the following is equivalent :

1. $\Gamma \rightarrow \Delta$ is finitary derivable
2. $\Gamma \rightarrow \Delta$ is derivable
3. $\Gamma \rightarrow \Delta$ cannot be falsified.

By the failure of finitary completeness of $L_{\kappa\omega}$, $\kappa > \omega_1$ it is also clear that the finitary analyzing theorem fails for $L_{\kappa\omega}$, $\kappa > \omega_1$.

The finitary completeness of $L_{\omega_1 \omega}$ is important in two ways :

1. Craigs interpolation theorem is true for finitary derivations. (See FEFERMAN 1968). Hence we have Craigs interpolation theorem for the semantical theory of $L_{\omega_1 \omega}$. (LOPEZ-ESCOBAR 1965).
2. The finitary derivations are definable in weak set theories. This is the essential step in the proof of completeness of L_A, A countable admissible. (BARWISE 1967).

Some of these themes are worth developing further, but since it is a sidetrack in this paper I will leave them here.

5. THE SKOLEM THEOREM

5.1. Given a sequent $\Gamma \to \Delta$, we now want to understand the function of the symbols in $\Gamma \to \Delta$ in generating classical trees over $\Gamma \to \Delta$. We take the function of the propositional connectives as straightforward. Our problem is to understand the quantifiers. First some preliminary definitions :

POSITIVE AND NEGATIVE

Positive and negative occurrences in a sequent $\Gamma \to \Delta$ are defined inductively by :

i) Any formula in Δ occurs positively in $\Gamma \to \Delta$;

ii) Any formula in Γ occurs negatively in $\Gamma \to \Delta$;

iii) If $\bigwedge F_i$ occurs positively (negatively in $\Gamma \to \Delta$ then each F_i occurs positively(negatively) in $\Gamma \to \Delta$;

iv) If $\neg F$ occurs positively (negatively) in $\Gamma \to \Delta$, then F occurs negatively (positively) in $\Gamma \to \Delta$; and

v) If $\forall xFx$ occurs positively(negatively) in $\Gamma \to \Delta$, then Fx occurs positively (negatively) in $\Gamma \to \Delta$.

GENERAL AND RESTRICTED

A quantifieroccurrence $\forall x$ in $\Gamma \to \Delta$ is general if it occurs as $\forall xFx$ with $\forall xFx$ positive in $\Gamma \to \Delta$. The occurrence is restricted if the $\forall xFx$ is negative.

The notions 'positive and negative occurrences' are quite common. 'General and restricted quantifiers' come from HERBRAND 1929.

The following lemma is obvious by inspecting the various cases (Quantifiers are formulaparts.)

LEMMA

If one quantifieroccurrence immediately succeds another (as formulaparts), then they are both general or both restricted quantifieroccurrences.

From the lemma :

THEOREM

For each strand of quantifieroccurrences in a classical tree T, all occurrences in it are general or they are all restricted.

The strands of quantifiers are either general or restricted depending on what the quantifieroccurrences are.

The general and the restricted quantifiers have different functions. In this chapter we will see that through the theory behind the Skolem theorem, we get control over the general quantifiers. With the Herbrand theorem in the next chapter we get control over also the restricted quantifiers.

5.2. Example.

We have the following secured classical tree over $\rightarrow \neg \forall x \neg \forall y\ [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)]$. The numbers above the arrows indicate the nodes.

$$Aa, Bb \overset{23}{\rightarrow} Bb$$

$$Aa \overset{21}{\rightarrow} Aa, Bb \qquad Aa \overset{22}{\rightarrow} \neg Bb, Bb$$

$$Aa \overset{20}{\rightarrow} Aa \wedge \neg Bb, Bb$$

$$Aa, \neg (Aa \wedge \neg Bb), \neg Bb \overset{19}{\rightarrow}$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb), Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc) \overset{18}{\rightarrow}$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb) \overset{17}{\rightarrow} \neg [Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), Ab \wedge \neg Ba \wedge \neg (Aa \wedge Bb) \overset{16}{\rightarrow} \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \overset{15}{\rightarrow} \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb)], \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \overset{14}{\rightarrow} \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)], \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \neg \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)] \overset{13}{\rightarrow} \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge Bb)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{12}{\rightarrow} \neg [Ab \wedge \neg Ba \wedge \neg (Aa \wedge \neg Bb)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{11}{\rightarrow} \forall y \neg [Ay \wedge \neg Ba \wedge \neg (Aa \wedge \neg By)]$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \neg \forall y \neg [Ay \wedge \neg Ba \wedge \neg (Aa \wedge \neg By)] \overset{10}{\rightarrow}$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{9}{\rightarrow}$$

$$Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba), \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{8}{\rightarrow}$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba) \overset{7}{\rightarrow}$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{6}{\rightarrow} \neg [Aa \wedge \neg Be \wedge \neg (Ae \wedge \neg Ba)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{5}{\rightarrow} \forall y \neg [Ay \wedge \neg Be \wedge \neg (Ae \wedge \neg By)]$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \neg \forall y \neg [Ay \wedge \neg Be \wedge \neg (Ae \wedge \neg By)] \overset{4}{\rightarrow}$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)], \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{3}{\rightarrow}$$

$$\forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)] \overset{2}{\rightarrow}$$

$$\overset{1}{\rightarrow} \neg \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)]$$

In the bottomsequent at node 1 we have a restricted quantifier $\forall x$ and a general quantifier $\forall y$. Corresponding to these we have one strand of restricted quantifiers $\forall x$ and one strand of general quantifiers $\forall y$.

The tree is a strong, secured, classical tree. It is not analyzing - we do not analyze $\forall x$ with respect to c. If we try

to do that, we must introduce a new parameter for
$\forall y \neg [Ay \wedge \neg Bc \wedge \neg (Ac \wedge \neg By)]$. The new parameter must then be used in an analysis of $\forall x$ etc. It is easy to see that there cannot be a finite analyzing classical tree over $\rightarrow \neg \forall x \neg \forall y \neg [Ay \wedge \neg Bx \wedge \neg (Ax \wedge \neg By)]$

The $\forall y$'s are used to introduce new parameters. In the $\forall x$'s we insert terms built up from e, (symbols from bottomsequent), and the new parameters. Whenever we have inserted a term for $\forall x$, we can introduce a new parameter for $\forall y$. We introduce new terms for $\forall x$ in 3,9,12. Corresponding to these we introduce new parameters in 5,11,16. That is we introduce c for y in 16 to get $\neg [Ac \wedge \neg Bb \wedge \neg (Ab \wedge \neg Bc)]$. This formulaoccurrence succeds $\neg \forall y \neg [Ay \wedge \neg Bb \wedge \neg (Ab \wedge \neg By)]$ in 13 which we have got by introducing b for x in 12. We can summarize :

In 3 e for x and then in 5 a for y.
In 9 a for x and then in 11 b for y.
In 12 b for x and then in 16 c for y.

The implied functional dependence, $e \rightarrow a$, $a \rightarrow b$, $b \rightarrow c$, will later be expressed with Skolem functions.

5.3. Transformations of trees.

I mentioned in 1.4 that the theory behind the Skolen theorem is developed as a theory of transformations of classical trees.

DEFINITION

A <u>classical morphism</u> $\mathfrak{M}$ is a transformation of classical trees into classical trees such that

i) For each classical tree T, T and $\mathfrak{M}(T)$ have the same treestructure, i.e. the same nodes and branches; and
ii) If the classical tree T_1 is an extension of the classical tree T_2, then $\mathfrak{M}(T_1)$ is an extension of $\mathfrak{M}(T_2)$.

DEFINITION

A <u>derivable morphism</u> is a classical morphism which transforms secured classical trees into secured classical trees. An <u>analyzing morphism</u> is a classical morphism which transforms analyzing trees into analyzing trees. A <u>falsifiability morphism</u> is a classical morphism which transforms analyzing not-secured trees into analyzing not-secured trees. A <u>classical isomorphism</u> is a classical morphism which is both a derivability and a falsifiability morphism.

The classical morphisms we use below will be simple. It will be clear that we preserve more than the tree-structure. Most of the rules are (almost) preserved. The transformations will also be such that all formulaoccurrences in a fibre are transformed in the same way. I do not think it worthwhile here to express these things with a sharper definition of classical morphism.

<u>5.4.</u> Our main example of a non-trivial classical isomorphism will be the Skolem morphism $\mathcal{S}$ defined below. For a one-sequent tree it coincides with the usual Skolem transformation. Skolem defined originally his transformation as a transformation to get rid of general quantifiers (See 1.1.) It is now more common to treat it as a transformation to get rid of restricted quantifiers. This is the natural thing to do when one treats the Skolem theorem semantically. We will follow Skolem, but the other way can be read into our treatment by stressing the falsifiability aspect of the theorem.

The functionsymbol f is called the Skolem function of Vx.

DEFINITION

$S(\Gamma \to \Delta)$ is obtained from $\Gamma \to \Delta$ by simultaneously carrying out the following operations for all general quantifiers $\forall x$:

i) erasing $\forall x$

ii) replacing all other occurrences of x in the range of $\forall x$ with $f(y_1,\cdots,y_N)$ where $\forall y_1,\cdots,\forall y_N$ are all the restricted quantifiers binding $\forall x$ $(N \geq 0)$ and f is an N-place functionsymbol not occurring in $\Gamma \to \Delta$ called the Skolemfunction of $\forall x$. Different general quantifiers have different Skolemfunctions.

$S(\Gamma \to \Delta)$ is called the Skolemtransform of $\Gamma \to \Delta$.

EXAMPLE

Consider the sequent $\to \forall x \neg \forall y \neg \forall z \neg \forall u R(x,y,z,u)$. If Skolemtransform is $\to \neg \forall y \neg \neg \forall u R(f,u,gy,u)$ where f is a 0-place and g a 1-place functionsymbol.

The Skolem transform S is well-defined up to the names of the Skolem functions. To make it well-defined we can either count sequent as equal when they are equal up to the names of the functionsymbols, or we can have a rule which picks out a unique Skolemfunction to each occurrence of a general quantifier in a sequent. Here we assume that one of these tactics is adopted and then disregard the problems about the names of the Skolemfunctions.

5.5. The Skolem morphism S.

The Skolem morphism is the natural extension of the Skolem transform to classical trees.

Assume that we have a classical tree T over the sequent $\Gamma \to \Delta$.

For each finite N let T_N be the subtree of T of all nodes of height $\leq N$. From the definition of classical tree we have:

LEMMA

T_N is a classical tree over $\Gamma \to \Delta$.

We introduce the following sets of terms :

$TERMS(e, T_N)$ = The set of terms built up from e and symbols in T_N.

$TERMS(e, S(\Gamma \to \Delta))$ = The set of terms built up from e and symbols in $S(\Gamma \to \Delta)$.

Below we shall define by recursion $\mathcal{S}(T_N)$ and

$\sigma_N : TERMS(e, T_N) \to TERMS(e, S(\Gamma \to \Delta))$.

The Skolem morphism $\mathcal{S}$ will be such that

i) each formulaoccurrence A at node μ in T is mapped into A^* at node μ in $\mathcal{S}(T)$ where A^* differs from A only in the erasure of general quantifiers and change of terms ;

ii) formulaoccurrences in the same fibre in T are changed into formulaoccurrences in the same fibre in $\mathcal{S}(T)$.

Both properties are obvious by the construction below.

The classical tree T_1 consists of $\Gamma \to \Delta$ alone. We define $\mathcal{S}(T_1)$ to be the classical tree consisting of $S(\Gamma \to \Delta)$.

Further $\mathrm{TERMS}(e,T_1) \subseteq \mathrm{TERMS}(e,S(\Gamma \to \Delta))$.

We define σ_1 to be the inclusionmap.

Now assume $\mathcal{S}(T_N)$ and σ_N defined. $(N \geq 1)$. Let μ be a node in T_{N+1} not in T_N. We now divide up into cases depending on which rule μ is a premiss of :

$\mathsf{M} \to$:

Say in T_{N+1} we have
$$\frac{\Gamma_1,\{A_{\alpha,\beta_{\alpha,\beta}}\}_{\alpha,\beta} \overset{\mu}{\to} \Delta_1}{\Gamma_1,\{\mathsf{M}_\beta\ A_{\alpha,\beta}\}_\alpha \to \Delta_1}$$

and in $\mathcal{S}(T_N)$
$$\Gamma_1^*,\{\mathsf{M}_\beta\ A^*_{\alpha,\beta}\}_\alpha \to \Delta_1^*$$

Then we put at μ in $\mathcal{S}(T_{N+1})$
$$\Gamma_1^*,\{A^*_{\alpha,\beta_{\alpha,\gamma}}\}_{\alpha,\gamma} \to \Delta_1^*$$

The structural rules and $\neg\to$, $\to\neg$, $\to\mathsf{M}$ are similar.

$\forall \to$:

Say in T_{N+1} we have
$$\frac{\Gamma_1,\{A_\alpha t_{\alpha,\beta}\}_{\alpha,\beta} \overset{\mu}{\to} \Delta_1}{\Gamma_1,\{\forall x_\alpha A_\alpha x_\alpha\}_\alpha \to \Delta_1}$$

and in $S(T_N)$
$$\Gamma_1^*,\{\forall x_\alpha A^*_\alpha x_\alpha\}_\alpha \to \Delta_1^*$$

Then since T_{N+1} is a classical tree, we have $t_{\alpha,\beta} \in \mathrm{TERMS}(e,T_N)$.

We put at μ in $S(T_{N+1})$
$$\Gamma_1^*,\{A^*_\alpha \sigma_N t_{\alpha,\beta}\}_{\alpha,\beta} \to \Delta_1^*$$

$\rightarrow \forall$:

Say in T_{N+1} we have $\dfrac{\Gamma_1 \rightarrow \{A_\alpha a_\alpha\}_\alpha, \Delta_1}{\Gamma_1 \rightarrow \{\forall x_\alpha A_\alpha x_\alpha\}_\alpha, \Delta_1}$

and in $S(T_N)$ $\qquad \Gamma_1^* \rightarrow \{A_\alpha^* t_\alpha^*\}_\alpha, \Delta_1^*$

We then put at μ in $S(T_{N+1})$ $\qquad \Gamma_1^* \rightarrow \{A_\alpha^* t_\alpha^*\}_\alpha, \Delta_1^*$

and $\sigma_{N+1}\, a_\alpha = t_\alpha^*$.

With this we have $S(T_{N+1})$ defined for all $\mu \in T_{N+1} \setminus T_N$. For $\mu \in T_N$ we let $\mathcal{S}^*$ behave on T_{N+1} as on T_N. We have defined $\mathcal{S}(T_{N+1})$. We get σ_{N+1} by

1. For $t \in \mathrm{TERMS}(e, T_N)$, $\sigma_{N+1}(t) = \sigma_N(t)$.
2. For parameters introduced by $\rightarrow \forall$ at nodes of level N+1 we get their σ_{N+1}-image as above.
3. For $f(t_1, \dots, t_m) \in \mathrm{TERMS}(e, T_{N+1})$
   $\sigma_{N+1}(f(t_1, \dots, t_m)) = f(\sigma_{N+1}(t_1), \dots, \sigma_{N+1}(t_m))$

We define $S(T)$ as the union of the $S(T_N)$'s.

From the definition of S, property ii) of S above, and property b of classical tree (see 3.7) we see that the same parameter introduced by $\rightarrow \forall$ has only one σ_{N+1} image. Then it is obvious that σ_{N+1} is a well-defined function. This is the place where we use property b of classical tree.

σ_{N+1} is an extension of σ_N.

Let TERMS(T) = The set of terms occurring in T.

We define τ_T: TERMS(T) $\rightarrow$ TERMS(e,S($\Gamma \rightarrow \Delta$)) by :

Let $t \in$ TERMS (T)

Then $t \in$ TERMS(e,T_N) for some N.

Put $\tau_T t = \sigma_N t$.

We have defined $\mathcal{S}$(T) and τ_T. In the following sections we will give properties of $\mathcal{S}$(T) and τ_T for various T. First we consider an example.

5.6. Example.

$$\underline{R(a,b,c,c),\ R(a,e,b,a),\ \forall uR(a,a,d,u)\ \overset{17}{\rightarrow}}$$

$$\underline{R(a,b,c,c),\ R(a,e,b,a)\ \overset{16}{\rightarrow}\ \forall uR(a,a,d,u)}$$

$$\underline{\forall uR(a,b,c,u),\ \forall uR(a,e,b,u)\ \overset{15}{\rightarrow}\ \forall uR(a,a,d,u)}$$

$$\underline{\forall uR(a,b,c,u),\ \forall uR(a,e,b,u)\ \overset{14}{\rightarrow}\ \forall z\neg\forall uR(a,a,z,u)}$$

$$\underline{\forall uR(a,b,c,u),\forall uR(a,e,b,u),\neg\forall z\neg\forall uR(a,a,z,u)\ \overset{13}{\rightarrow}}$$

$$\underline{\neg\forall z\neg\forall uR(a,a,z,u),\forall uR(a,b,c,u),\forall uR(a,e,b,u)\ \overset{12}{\rightarrow}}$$

$$\underline{\neg\forall z\neg\forall uR(a,a,z,u)\ \overset{11}{\rightarrow}\ \neg\forall uR(a,b,c,u),\neg\forall uR(a,e,b,u)}$$

$$\underline{\neg\forall z\neg\forall uR(a,a,z,u)\ \overset{10}{\rightarrow}\ \forall z\neg\forall uR(a,b,z,u),\neg\forall uR(a,e,b,u)}$$

$$\underline{\neg\forall z\neg\forall uR(a,a,z,u),\neg\forall z\neg\forall uR(a,b,z,u)\ \overset{9}{\rightarrow}\ \neg\forall uR(a,e,b,u)}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u),\forall y\neg\forall z\neg\forall uR(a,y,z,u)\ \overset{8}{\rightarrow}\ \neg\forall uR(a,e,b,u)}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u),\ \forall y\neg\forall z\neg\forall uR(a,y,z,u)\ \overset{7}{\rightarrow}\ \forall z\neg\forall uR(a,e,z,u)}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u)\ \overset{6}{\rightarrow}\ \forall z\neg\forall uR(a,e,z,u)}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u),\ \neg\forall z\neg\forall uR(a,e,z,u)\ \overset{5}{\rightarrow}}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u),\ \forall y\neg\forall z\neg\forall uR(a,y,z,u)\ \overset{4}{\rightarrow}}$$

$$\underline{\forall y\neg\forall z\neg\forall uR(a,y,z,u)\ \overset{3}{\rightarrow}}$$

$$\underline{\overset{2}{\rightarrow}\ \neg\forall y\neg\forall z\neg\forall uR(a,y,z,u)}$$

$$\overset{1}{\rightarrow}\ \forall x\neg\forall y\neg\forall z\neg\forall uR(x,y,z,u)$$

In $\rightarrow \forall x\neg\forall y\neg\forall z\neg\forall uR(x,y,z,u)$ there are two general quantifiers, $\forall x$ and $\forall z$. Let f be the o-place Skolemfunction of $\forall x$ and g the 1-place Skolemfunction of $\forall z$. We write T for the classical tree above and $\Gamma \rightarrow \Delta$ for the bottomsequent $\rightarrow \forall x\neg\forall y\neg\forall z\neg\forall ur(x,y,z,u)$. So $S(\Gamma \rightarrow \Delta) = \rightarrow\neg\forall y\neg\neg\forall uR(f,y,gy,u)$.

Then $\mathcal{S}(T)$ is :

$$\underline{R(f,ge,gge,gge),R(f,e,ge,f),\forall uR(f,f,gf,u) \overset{17}{\rightarrow}}$$

$$\underline{R(f,ge,gge,gge),R(f,e,ge,f) \overset{16}{\rightarrow}}$$

$$\underline{\forall uR(f,ge,gge,u),\forall uR(f,e,ge,u) \overset{15}{\rightarrow}}$$

$$\underline{\forall uR(f,ge,gge,u),\forall uR(f,e,ge,u) \overset{14}{\rightarrow}}$$

$$\underline{\forall uR(f,ge,gge,u),\forall uR(f,e,ge,u),\neg\neg\forall uR(f,f,gf,u) \overset{13}{\rightarrow}}$$

$$\underline{\neg\neg\forall u\ (f,f,gf,u),\forall uR(f,ge,gge,u),\forall uR(f,e,ge,u) \overset{12}{\rightarrow}}$$

$$\underline{\neg\neg\forall uR(f,f,gf,u) \overset{11}{\rightarrow} \forall uR(f,ge,gge,u),\neg\forall uR(f,e,ge,u)}$$

$$\underline{\neg\neg\forall uR(f,f,gf,u) \overset{10}{\rightarrow} \forall uR(f,ge,gge,u),\neg\forall uR(f,e,ge,u)}$$

$$\underline{\neg\neg\forall uR(f,f,gf,u),\neg\neg\forall uR(f,ge,gge,u) \overset{9}{\rightarrow} \neg\forall uR(f,e,ge,u)}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u),\forall y\neg\neg\forall uR(f,y,gy,u) \overset{8}{\rightarrow} \neg\forall uR(f,e,ge,u)}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u),\forall y\neg\neg\forall uR(f,y,gy,u) \overset{7}{\rightarrow} \neg\forall uR(f,e,ge,u)}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u) \overset{6}{\rightarrow} \neg\forall uR(f,e,ge,u)}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u),\neg\neg\forall uR(f,e,ge,u) \overset{5}{\rightarrow}}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u),\forall y\neg\neg\forall uR(f,y,gy,u) \overset{4}{\rightarrow}}$$

$$\underline{\forall y\neg\neg\forall uR(f,y,gy,u) \overset{3}{\rightarrow}}$$

$$\underline{\overset{2}{\rightarrow} \neg\forall y\neg\neg\forall uR(f,y,gy,u)}$$

$$\overset{1}{\rightarrow} \neg\forall y\neg\neg\forall uR(f,y,gy,u)$$

Further we have

$$\text{TERMS }(T) = \{e,a,b,c,d\}$$

$$\text{TERMS }(e,S(\Gamma \rightarrow \Delta)) = \{e,f,ge,gf,gge,ggf,\dots\}$$

and $\tau_T e = e$, $\tau_T a = f$, $\tau_T b = ge$, $\tau_T c = gge$, $\tau_T d = gf$.

5.7. Properties of $\mathcal{S}$ and τ_T.

THEOREM $\mathcal{S}$ is a classical morphism.

Proof: From the definition of $\mathcal{S}$ it is obvious that $\mathcal{S}$ and $\mathcal{S}(T)$ have the same tree structure and if T_1 is an extension of T_2, then $\mathcal{S}(T_1)$ is an extension of $\mathcal{S}(T_2)$.

Let T be a classical tree over $\Gamma \to \Delta$.

It remains to be proved that $\mathcal{S}(T)$ is a classical tree.

$\mathcal{S}(T)$ is obviously a tree of sequents.

Since $\mathcal{S}(T)$ does not contain any general quantifiers, condition b in the definition of classical tree is trivially satisfied.

All the terms in $\mathcal{S}(T)$ are contained in $\mathrm{TERMS}(e, S(\Gamma \to \Delta))$, and hence condition a in the definition is satisfied. Ω

Observe that if T is a classical tree over $\Gamma \to \Delta$ the $\mathcal{S}(T)$ is a classical tree over $S(\Gamma \to \Delta)$.

The following lemma is obvious by the construction :

LEMMA

Let T be a classical tree over $\Gamma \to \Delta$, and $A(t_1,\cdots,t_N)$ a formulaoccurrence in T without any general variables. $t_1,\cdots,t_N$ are all the terms occurring in $A(t_1,\cdots,t_N)$. Then in applying $\mathcal{S}$ $A(t_1,\cdots,t_N)$ is changed to $A(\tau t_1,\cdots,\tau t_N)$.

THEOREM

$\mathcal{S}$ is a derivability morphism.

Proof: Let T be a secured classical tree.

Let Γ_1, $A(t_1,\cdots,t_N) \to A(t_1,\cdots t_N)$, Δ_1 be an axiom occurring in T with terms $t_1,\dots,t_N$.

From the lemma we get by applying $\mathcal{S}$ that the sequent is changed to

$$\Gamma_1^*, A(\tau t_1,\dots,\tau t_N) \to A(\tau t_1,\dots,\tau t_N), \Delta_1^*$$

which is also an axiom.

Hence $\mathcal{S}(T)$ is secured. Ω

This theorem shows that if $\vdash \Gamma \to \Delta$ then also $\vdash S(\Gamma \to \Delta)$. This is the easy part of the Skolem theorem - the part that can be generalized to intuitionistic logic and modal logics. For the other part of the Skolem theorem we need a finer discussion of the properties of $\mathcal{S}$ and τ_T.

For later reference we note that in the proofs below (and also the proof above) that we only need the following properties of axioms in $L_{\kappa\omega}$:

PROPERTY 1

If $\Gamma \to \Delta$ is an axiom, Γ^* contains at least the same atomic formulas as Γ, and, Δ^* at least the same as Δ, then $\Gamma^* \to \Delta^*$ is an axiom.

PROPERTY 2

If $\Gamma \to \Delta$ is an axiom, σ a mapping of terms into terms and $\Gamma^* \to \Delta^*$ is got from $\Gamma \to \Delta$ by applying σ on all the terms, then $\Gamma^* \to \Delta^*$ is an axiom.

PROPERTY 3

If $\Gamma \to \Delta$ is an axiom, then there is a finite subsequent $\Gamma_0 \to \Delta_0$ of $\Gamma \to \Delta$ which is also an axiom.

Now we come to the further lemmas of $\mathcal{S}$ and τ_T.

For the rest of this section let T be a classical tree over $\Gamma \to \Delta$.

LEMMA 1

If $\Gamma \to \Delta$ contains at least one restricted quantifier, and T is analyzing, then all terms which can be built up from e and symbols in T do actually occur in T.

Proof: Assume that $\Gamma \to \Delta$ contains one restricted quantifier and that T is analyzing.

Let t be a term built up from e and symbols in T.

Then in some branch β in T a formula $\forall xAx$ must occur in an antecedent.

Since β is analyzing, Ft occurs in β. Hence t occurs in β. Ω

LEMMA 2

If $\Gamma \to \Delta$ contains at least one restricted quantifier and T is analyzing, then the function τ_T is onto.

Proof: Assume that $\Gamma \to \Delta$ contains one restricted quantifier and that T is analyzing.

Assume τ not onto.

Let $u \in \mathrm{TERMS}(e, S(\Gamma \to \Delta))$ be not in the τ_T-image and u such a term of minimal length.

The lemma is proved by deriving contradictions in the following 3 cases :

i) u does not contain any Skolemfunctions. Then u is a term built up from symbols in T and e.
By lemma 1 $u \in \mathrm{TERMS}(T)$.
But in this case $\tau_T u = u$ and we have a contradiction.

ii) $u = f(u_1,\dots,u_N)$ where f is a functionsymbol occurring in $\Gamma \to \Delta$.
By choice of u there exists $t_1,\dots,t_N \in \mathrm{TERMS}(T)$ with $\tau_T u = t_1,\dots,\tau_T u_N = t_N$.
By lemma 1 $f(t_1,\dots,t_N) \in \mathrm{TERMS}(T)$.
Now $\tau_T f(t_1,\dots,t_N) = f(u_1,\dots,u_N) = u$.
Contradiction.

iii) $u = f(u_1,\dots,u_N)$ where f is a Skolemfunction $(N \geq o)$.
By choice of u $\tau_T t_1 = u_1,\dots,\tau_T t_N = u_N$ for some $t_1,\dots,t_N \in \mathrm{TERMS}(T)$ f is a Skolemfunction of a general quantifier $\forall x$ in T. Since T is analyzing, somewhere in T $\forall x$ must occur as formula $\forall x F x$ with restricted analysis $t_1,\dots,t_N$.
For this $\forall x$ we introduce parameter a. Then $a \in \mathrm{TERMS}(T)$ and $\tau_T a = f(\tau t_1,\dots,\tau t_N) = u$. Contradiction. Ω

THEOREM

$\mathcal{S}$ is an analyzing morphism.

Proof: Let T be an analyzing classical tree.

Let β be a branch in $\mathcal{S}(T)$. We must prove that β is analyzing. (See 4.2). By comparing β in $\mathcal{S}(T)$ with β in T all the cases in the definition are obvious except that instead of iv). We prove:

If $\forall xFx$ occurs in an antecedent in β in $\mathcal{S}(T)$, then for every term t <u>occurring in $\mathcal{S}(T)$,</u> Ft occurs in β in $\mathcal{S}(T)$ as a successor to the fibre of $\forall xFx$.

Instead of "occuring in $\mathcal{S}(T)$" we must have "in TERMS(e,s($\Gamma \to \Delta$))".

If there are no restricted quantifier in $S(\Gamma \to \Delta)$, then it is trivially true.

Assume then there is a restricted quantifier in $S(\Gamma \to \Delta)$. Then there is also one in $\Gamma \to \Delta$.

From lemma 2 : τ_T is onto.

Let $u \in$ TERMS(e,s($\Gamma \to \Delta$)).

There is $t \in$ TERMS(T) with $\tau_T t = u$. From lemma 1 t occurs in T. But then also $\tau_T t$ occurs in S(T). And TERMS ($\mathcal{S}$(T)) = TERMS(e,s($\Gamma \to \Delta$)). Hence iv) is true for β also in this case. β is analyzing in $\mathcal{S}(T)$. $\mathcal{S}(T)$ is analyzing. Ω

The next step is to prove that $\mathcal{S}$ is a falsifiability morphism. Here we need the strong analyzing theorem.

First we note that in a strong classical tree "same restricted analysis" implies "same analysis".

LEMMA 3

Assume T is a strong classical tree and F_1, F_2 two formula-occurrences in T in the same strand of formulaparts. If F_1 and F_2 have the same restricted analysis, then they have the same analysis.

An example will make the lemma clear.

Say F_1 has analysis $\langle\langle x_1,t_1\rangle,\langle y_1,a_1\rangle,\langle x_2,t_2\rangle\rangle$ and F_2 $\langle\langle x_1,t_1\rangle, \langle y_1,b_1\rangle,\langle x_2,t_2\rangle\rangle$. Here the x's are the restricted variables and y_1 the general variable.

We must prove that $a_1 = b_1$. We assume that the quantifiers occur in the order we have written them i.e. $\forall x_1$ is outermost, then $\forall y_1$, and $\forall x_2$.

Now $\forall y_1$ occurs as $\forall y_1 G$ both as predecessor to F_1 and as predecessor to F_2; and both occurrences are in the same fibre. Since T is strong we must use the same parameter in analyzing the occurrences of $\forall y_1 G$. We conclude that $a_1 = b_1$.

The general case of the lemma is proved in the same way.

LEMMA 4

If T is a strong classical tree, then the function τ_T us 1-1.

Proof: Let T be a strong classical tree. Assume that τ_T is not 1-1. Let $u \in \text{TERMS}(e,s(\Gamma \to \Delta))$ be such that $\tau_T t' = \tau_T t'' = u$, $t' \neq t''$ and u such a term of minimal length. We divide up into three cases :

i) u does not contain any Skolemfunction.

Then $t' = u$ and $t'' = u$. Contradiction.

ii) $u = f(u_1,\dots,u_n)$ where f is a functionsymbol from $\Gamma \to \Delta$.

Then $t_1 = \tau_T^{-1}u_1,\dots, t_n = \tau_T^{-1}u_n$ are well-defined,

and $t' = f(t_1,\dots,t_n) = t''$. Contradiction.

iii) $u = f(u_1,\dots,u_n)$ where f is a Skolemfunction $(n \geq o)$.

Then t' and t'' must be parameters introduced by $\to \forall$ in T.

Say t' is introduced from F' and t'' from F''.

F' and F'' are in the same strand of formulaparts.

By choice of u $t_1 = \tau_T^{-1}u_1, \ldots, t_n = \tau_T^{-1}u_n$ are well-defined.

The restricted analysis of both F' and F'' are given by $t_1, \ldots, t_n$. Hence they have the same restricted analysis. By lemma 3 they have also the same analysis. Since T is strong, we must have $t' = t''$. Contradiction. Ω

LEMMA 5

Let T be a strong analyzing classical tree. Then $\mathcal{S}$ transforms not-secured branches in T into not-secured branches in $\mathcal{S}(T)$.

Proof: Assume that β is a secured branch in $\mathcal{S}(T)$. Let ν be a secured node in β in $\mathcal{S}(T)$. Then at ν in $\mathcal{S}(T)$ we can assume that we have a sequent

$$\Gamma_1^*, \Gamma_2^* \rightarrow \Delta_2^*, \Delta_1^*$$

where

Γ_2^*, Δ_2^* consists of finitely many atomic formulas

$\Gamma_2^* \rightarrow \Delta_2^*$ is an axiom.

Then at ν in T we will have atomic formulas corresponding to $\Gamma_2{}^*$, while corresponding to the formulas in $\Delta_2{}^*$ we may have atomic formulas prefixed by a parameter. Since Δ_2^* is finite and β is analyzing, then there must be a node μ in β above ν where all the quantifiers in the formulas corresponding to Δ_2^* are analyzed away. Hence at μ in $\mathcal{S}(T)$ we have

$$\Gamma_3^*, \Gamma_2^* \rightarrow \Delta_2^*, \Delta_3^*$$

while at μ in T

$$\Gamma_3, \Gamma_2 \rightarrow \Delta_2, \Delta_3 \ .$$

Here $\Gamma_2^* \rightarrow \Delta_2^*$ is an axiom.

Γ_2, Γ_2^*, Δ_2,Δ_2^* consists of atomic formulas. $\Gamma_2^* \to \Delta_2^*$ is got from $\Gamma_2 \to \Delta_2$ by applying τ_T on the terms.

By lemma 4 τ_T is 1-1.

Hence we get $\Gamma_2 \to \Delta_2$ from $\Gamma_2^* \to \Delta_2^*$ by a mapping of terms into terms.

Therefore $\Gamma_2 \to \Delta_2$ is an axiom and μ is secured in T.

β is secured in T. Ω

THEOREM

$\mathcal{S}$ is a falsifiability morphism.

Proof: Let T be an analyzing not-secured tree over $\Gamma \to \Delta$. We shall prove that $\mathcal{S}(T)$ is an analyzing not-secured tree over $S(\Gamma \to \Delta)$.

By the theorem above $\mathcal{S}(T)$ is an analyzing tree over $S(\Gamma \to \Delta)$. We use the strong analyzing theorem (4.6) to get a strong analyzing tree T^* over $\Gamma \to \Delta$.

Since T is not-secured analyzing we get by the corollary to the completeness theorem (4.5) that T^* is not-secured. By the lemma above $\mathcal{S}(T^*)$ is a not-secured analyzing tree over $S(\Gamma \to \Delta)$.

Again by the corollary we conclude that $\mathcal{S}(T)$ is not-secured. Ω

The theorems proved are put together to get

THEOREM

$\mathcal{S}$ is a classical isomorphism.

As a straightforward corollary

COROLLARY 1

For any sequent $\Gamma \to \Delta$: $\vdash \Gamma \to \Delta$ if and only if $\vdash S(\Gamma \to \Delta)$.

COROLLARY 2

Let $\Gamma^* \to \Delta^*$ be a sequent got from $\Gamma \to \Delta$ by introducing Skolemfunctions for some but not necessarily all general quantifiers. Then $\vdash \Gamma \to \Delta$ if and only if $\vdash \Gamma^* \to \Delta^*$.

Proof: By assumption $S(\Gamma^* \to \Delta^*) = S(\Gamma \to \Delta)$ and we are done by corollary 1. Ω

COROLLARY 3

Let $S(\Gamma \to \Delta) = \Gamma_1 \to \Delta_1$. Then $\vdash \Gamma_1, \Delta \to \Gamma, \Delta_1$.

Proof: Since $\vdash \Gamma, \Delta \to \Gamma, \Delta$, we can use corollary 2 to get the result. Ω

5.9. Finitary derivations.

We saw in 4.7 that finitary derivations are important for countable languages. Now we note that the Skolem theory can easily be carried over into finitary derivations.

THEOREM

$\mathcal{S}$ transforms finitary classical trees into finitary classical trees.

Proof: Obvious since in applying $\mathcal{S}$ we do not change the number of formulas at the nodes. Ω

Hence

THEOREM

$\mathcal{S}$ transforms

i) secured finitary trees into secured finitary trees;

ii) analyzing finitary trees into analyzing finitary trees; and

iii) not-secured analyzing finitary trees into not-secured analyzing finitary trees.

So seemingly we have developed the Skolem theory for finitary classical trees for all $L_{\kappa\omega}$. But it is only in conjunction with an analyzing theorem that the theorem above becomes important.

It is an easy exercise now to develop the Skolem theory for all admissible languages; but it is only in the admissible languages $L_A \subseteq L_{\omega,\omega}$ that we can conclude: For any A-finite sequent $\Gamma \to \Delta$: There is an A-finite derivation of $\Gamma \to \Delta$ if and only if there is an A-finite derivation of $S(\Gamma \to \Delta)$.

6. THE HERBRAND THEOREM.

6.1. With the Skolem theory we have got the general quantifiers under control. Now we turn to the restricted quantifiers. First we observe that in $L_{\omega\omega}$ we cannot have a classical isomorphism $\mathfrak{M}$ which gives $\mathfrak{M}(\Gamma \to \Delta)$ recursively from $\Gamma \to \Delta$ with $\mathfrak{M}(\Gamma \to \Delta)$ without restricted quantifiers. For then we would have a decision procedure for $L_{\omega\omega}$. In $L_{\omega\omega}$ we get something close to a classical isomorphism. We get a sequence $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \dots$ of falsifiability morphisms which "in the limit" is a derivability morphism.

For languages stronger than $L_{\omega\omega}$ we get classical isomorphisms. The theory below is developed directly for $L_{\kappa\omega}$.

6.2. Herbrand domains, Herbrand transforms, and Herbrand morphisms.

DEFINITION

A sequence $\{t_\alpha\}_\alpha$ of terms is an Herbrand domain in $L_{\kappa\omega}$ if for every set S of symbols with cardinality $S < \kappa$, the subsequence of $\{t_\alpha\}_\alpha$ of terms built up from symbols in S is of length $< \kappa$.

We have the following examples of Herbrand domains :

EXAMPLE 1

Let $\mathfrak{D}_n$ be a sequence without repetition of all terms of length $< n$. $\mathfrak{D}_n$ is an Herbrand domain in all $L_{\kappa\omega}$.

EXAMPLE 2

Let $\mathfrak{D}$ be a sequence without repetition of all terms. $\mathfrak{D}$ is not an Herbrand domain in $L_{\omega\omega}$, but for each $\kappa > \omega$ is an Herbrand domain in $L_{\kappa\omega}$.

Consider the admissible language L_A. In this language we define the Herbrand domains as above except that we put A-finite instead of

$< \kappa$. Let ε be an A-recursive sequence without repetition of all terms in the language of L_A. Then if $L_A \neq L_{\omega\omega}$, ε is an Herbrand domain in L_A.

DEFINITION

Let $\mathfrak{D}$ be an Herbrand domain. The $\mathfrak{D}$-Herbrand transform, $H_{\mathfrak{D}}$, is defined by :

For any sequent $\Gamma \to \Delta$ let $\mathfrak{D}_o$ be the subsequence of $\mathfrak{D}$ of terms built up from e and symbols in $S(\Gamma \to \Delta)$. We then get $H_{\mathfrak{D}}(\Gamma \to \Delta)$ from $\Gamma \to \Delta$ by first applying S and then exchanging each (restricted) quantifier $\forall x$ with $\bigwedge\limits_{x \in \mathfrak{D}_o}$.

The $\mathfrak{D}$ -Herbrand transform has the following natural extension to classical trees.

DEFINITION

Let $\mathfrak{D}$ be an Herbrand domain. The $\mathfrak{D}$-Herbrand morphism, $\mathcal{H}_{\mathfrak{D}}$, is defined by :

For any classical tree T over $\Gamma \to \Delta$ let $\mathfrak{D}_o$ be the subsequence of $\mathfrak{D}$ of terms built up from e and symbols in $S(\Gamma \to \Delta)$. We then get $\mathcal{H}_{\mathfrak{D}}(T)$ by first applying S and then by :

1) Erasing all formulaoccurrences with a term not in $\mathfrak{D}_o$ in its analysis.

2) Replacing each (restricted) quantifier $\forall x$ with $\bigwedge\limits_{x \in \mathfrak{D}_o}$.

6.3. Example.

We continue our example from section 5.6. There we considered a classical tree T over

$$\Gamma \to \Delta = \to \forall x \neg \forall y \neg \forall z \neg \forall u R(x,y,z,u)$$

We construct the classical tree $\mathcal{S}(T)$ over

$$S(\Gamma \to \Delta) = \to \neg \forall y \neg\neg \forall u R(f,y,gy,u)$$

As Herbrand domain we take a sequence without repetition of all terms in our language of length ≤ 1. Call it $\mathcal{D}_1$.
The subsequence of terms built up from e and symbols in $S(\Gamma \to \Delta)$ consists of two terms, e and f. We write for this subsequence $\langle e,f \rangle$.

To construct $\mathcal{H}_{\mathcal{D}_1}(T)$ we first observe that

$$H_{\mathcal{D}_1}(\Gamma \to \Delta) = \to \neg \bigwedge_{y \in \langle e,f \rangle} \neg\neg \bigwedge_{u \in \langle e,f \rangle} R(f,y,gy,u).$$

We must now start with $\mathcal{S}(T)$ and erase formulaoccurrences which have a term different from e and f in their analysis. These formulaoccurrences are

in 9	$\neg\neg \forall u R(f,ge,gge,u)$
in 10,11	$\neg \forall u R(f,ge,gge,u)$
in 12,13,14,15	$\forall u R(f,ge,gge,u)$
in 16,17	$R(f,ge,gge,gge)$

We then get $\mathcal{H}_{\mathcal{D}_1}(T)$:

$$R(f,e,ge,f),\ \bigwedge_{u\in\langle e,f\rangle} R(r,r,gf,u) \overset{17}{\rightarrow}$$

$$R(f,e,ge,f) \overset{16}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u)$$

$$\bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u) \overset{15}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u)$$

$$\bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u) \overset{14}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u)$$

$$\bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u),\ \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u) \overset{13}{\rightarrow}$$

$$\neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u),\ \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u) \overset{12}{\rightarrow}$$

$$\neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u) \overset{11}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u) \overset{10}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,f,gf,u) \overset{9}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u),\ \bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u) \overset{8}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u),\ \bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u) \overset{7}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u) \overset{6}{\rightarrow} \neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u)$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u),\ \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,e,ge,u) \overset{5}{\rightarrow}$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u)\ \ \bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u) \overset{4}{\rightarrow}$$

$$\bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u) \overset{3}{\rightarrow}$$

$$\overset{2}{\rightarrow} \neg \bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u)$$

$$\overset{1}{\rightarrow} \neg \bigwedge_{y\in\langle e,f\rangle} \neg\neg \bigwedge_{u\in\langle e,f\rangle} R(f,y,gy,u)$$

<u>6.4.</u> Now we will prove that $\mathscr{H}_{\mathscr{D}}$ is a classical morphism.

LEMMA 1

For each Herbrand domain $\mathscr{D}$ and each sequent $\Gamma \to \Delta$, the $\mathscr{D}$-Herbrand morphism of the one-sequent tree $\Gamma \to \Delta$ is equal to the one-sequent tree of $H_{\mathscr{D}}(\Gamma \to \Delta)$. In symbols

$$\mathscr{H}_{\mathscr{D}}(\Gamma \to \Delta) = H_{\mathscr{D}}(\Gamma \to \Delta).$$

Proof: In a one-sequent tree the analysis of any formula-occurrence is the empty sequent. So in the construction of $\mathscr{H}_{\mathscr{D}}(\Gamma \to \Delta)$ we do not erase any formulaoccurrences. In the construction of $\mathscr{H}_{\mathscr{D}}(\Gamma \to \Delta)$ we only replace quantifiers with conjunctions. Clearly $\mathscr{H}_{\mathscr{D}}(\Gamma \to \Delta) = H_{\mathscr{D}}(\Gamma \to \Delta)$. Ω

LEMMA 2

For each $\mathscr{D}, T$ $\mathscr{H}_{\mathscr{D}}(T)$ has the same tree structure as T.

Note that the question whether we erase a formulaoccurrence F in $\mathscr{S}(T)$ in constructing $\mathscr{H}_{\mathscr{D}}(T)$ depends only on the formula-occurrences preceding it in $\mathscr{S}(T)$. From this :

LEMMA 3

For any $\mathscr{D}$ and any T_1, T_2 with T_1 an extension of T_2 we have that $\mathscr{H}_{\mathscr{D}}(T_1)$ is an extension of $\mathscr{H}_{\mathscr{D}}(T_2)$.

THEOREM

For any Herbrand domain $\mathscr{D}$, $\mathscr{H}_{\mathscr{D}}$ is a classical morphism. If T is a classical tree over $\Gamma \to \Delta$, then $\mathscr{H}_{\mathscr{D}}(T)$ is a classical tree over $H_{\mathscr{D}}(\Gamma \to \Delta)$.

Proof: Using the lemmas above and that $\mathcal{H}_{\mathfrak{D}}(T)$ does not contain any quantifiers we only need to prove that $\mathcal{H}_{\mathfrak{D}}(T)$ is a tree of sequents whenever T is a classical tree.

So assume T is a classical tree.

By the previous chapter $\mathcal{S}(T)$ is a classical tree. We must show that instances of the rules in $\mathcal{S}(T)$ are carried over into instances of the rules in $\mathcal{H}_{\mathfrak{D}}(T)$.

As example consider the following instance of $\forall \rightarrow$ in $\mathcal{S}(T)$:

$$\frac{\Gamma_1, Fs, Gt, Hu \rightarrow \Delta_1}{\Gamma_1, \forall xFx, \forall yGy, \forall zHz \rightarrow \Delta_1}$$

Say that $\forall xFx$ has a term not in $\mathfrak{D}$ in its analysis and that $\forall yGy$ and $\forall zHz$ have only terms from $\mathfrak{D}$ in its analysis. Say further that $t \notin \mathfrak{D}$ and $u \in \mathfrak{D}$. Then in $\mathcal{H}_{\mathfrak{D}}(T)$ we get :

$$\frac{\Gamma_1^*, H^*u \rightarrow \Delta_1^*}{\Gamma_1^*, \bigwedge_{y \in \mathfrak{D}_0} Gy, \bigwedge_{z \in \mathfrak{D}_0} H^*z \rightarrow \Delta_1^*}$$

and we have an instance of $\bigwedge \rightarrow$. Instances of the other rules are even easier to treat. Ω

<u>6.5.</u> We now turn to the special properties of $\mathcal{H}_{\mathfrak{D}}$.

Observe that for atomic formulas in $\mathcal{S}(T)$ in going to $\mathcal{H}_{\mathfrak{D}}(T)$ we either erase them or let them be as they are. The following lemma is then obvious :

LEMMA

Let $\mathcal{D}_1, \mathcal{D}_2$ be Herbrand domain with $\mathcal{D}_1 \subseteq \mathcal{D}_2$. For any classical tree T and any node ν:

The atomic formulas in succedent (antecedent) at ν in $\mathcal{H}_{\mathcal{D}_1}(T)$

$\subseteq$ The atomic formulas in succedent (antecedent) at ν in $\mathcal{H}_{\mathcal{D}_2}(T)$

$\subseteq$ The atomic formulas in succedent (antecedent) at ν in $\mathcal{S}(T)$.

Since the question of whether a node is secured concerns only the atomic formulas :

LEMMA

Let $\mathcal{D}$ be an Herbrand domain and T a classical tree. If ν is a not-secured node in $\mathcal{S}(T)$, then ν is also not-secured in $\mathcal{H}_{\mathcal{D}}(T)$.

For the next theorem we note that $\mathcal{H}_{\mathcal{D}}$ (like $\mathcal{S}$) behaves in the same way for formulaoccurences in the same fibre.

THEOREM

For any Herbrand domain $\mathcal{D}$, $\mathcal{H}_{\mathcal{D}}$ is an analyzing morphism.

Proof: Let T be an analyzing tree.

By previous chapter $\mathcal{S}(T)$ is also analyzing.

Let β be a branch. We know that β is analyzing in $\mathcal{S}(T)$ and shall show that it is also analyzing in $\mathcal{H}_{\mathcal{D}}(T)$. The proof is divided up into two cases corresponding to the definition of analyzing branch. (4.3)

We treat the important case :

Say that $\bigwedge\limits_{x \in \mathcal{D}_0} Fx^*$ occurs in an antecedent of β in $\mathcal{H}_{\mathcal{D}}(T)$ and that the corresponding formulaoccurrence in $\mathcal{S}(T)$ is $\forall x Fx$.

Let t be a term in $\mathfrak{D}_0$. Since $\mathcal{S}(T)$ is analyzing and contains a restricted quantifier, t must occur in $\mathcal{S}(T)$. So Ft must occur in β as a successor to the fibre of $\forall xFx$ in $\mathcal{S}(T)$.

Correspondingly, F^*t must occur in β in $\mathcal{H}_{\mathfrak{D}}(T)$ as a successor to the fibre of $\bigwedge\!\!\!\bigwedge_{x \in \mathfrak{D}_0} F^*x$.

The other cases are treated similarly. Ω

THEOREM

For any Herbrand domain $\mathfrak{D}$, $\mathcal{H}_{\mathfrak{D}}$ is a falsifiability morphism.

Proof: Let T be an analyzing not-secured tree.

Then $\mathcal{S}(T)$ is not-secured, analyzing tree. From the lemma above $\mathcal{H}_{\mathfrak{D}}(T)$ is not-secured. From the theorem $\mathcal{H}_{\mathfrak{D}}(T)$ is analyzing. Ω

In the introduction to this chapter we gave an argument which shows that for no Herbrand domain in $L_{\omega\omega}$ $\mathcal{H}_{\mathfrak{D}}$ is a derivability morphism (and hence a classical isomorphism). It turns out that we can almost get $\mathcal{H}_{\mathfrak{D}}$ a derivability morphism.

THEOREM

(For all $L_{\kappa\omega}$). For Herbrand domains $\mathfrak{D}_1, \mathfrak{D}_2$ with $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$ and T a classical tree, if $\mathcal{H}_{\mathfrak{D}_1}(T)$ is secured, then also $\mathcal{H}_{\mathfrak{D}_2}(T)$.

LEMMA

(For all $L_{\kappa\omega}$). For T a classical tree and N a finite set of nodes of T which only contains finite sequents there is a finite Herbrand domain $\mathfrak{D}$ such that for all nodes ν in N :

The atomic formulas in succedent (antecedent) at ν in $\mathcal{H}_{\mathfrak{D}}(T)$
= The atomic formulas in succedent (antecedent) at ν in $\mathcal{S}(T)$.

Proof: Take $\mathfrak{D}$ to be a sequence without repetition of all terms contained in analysis of formulaoccurrences in nodes in N in $\mathcal{S}(T)$.

The proof is then obvious. Ω

THEOREM

Let T be a secured finitary tree in $L_{\omega\omega}$ (i.e. a secured tree in $L_{\omega\omega}$ with only finite branchings in the trivial rule). There is then a finite Herbrand domain $\mathfrak{D}$ with $\mathcal{H}_{\mathfrak{D}}(T)$ secured.

Proof: $\mathcal{S}(T)$ is also a finitary secured tree in $L_{\omega\omega}$. There is then a finite set N of secured nodes in $\mathcal{S}(T)$ such that any branch in $\mathcal{S}(T)$ contains a node in N.

Choose $\mathfrak{D}$ to be the finite Herbrand domain given from N by the lemma above. We then get the nodes in N secured in $\mathcal{H}_{\mathfrak{D}}(T)$ and hence $\mathcal{H}_{\mathfrak{D}}(T)$ secured. Ω

6.6. Some particular Herbrand domains.

6.6.1. For each natural number n let $\mathfrak{D}_n$ be a sequence without repetition of all terms of length $\leq n$. $\mathfrak{D}_n$ is obviously a Herbrand domain in all $L_{\kappa\omega}$. We put $\mathcal{H}_n = \mathcal{H}_{\mathfrak{D}_n}$, $H_n = H_{\mathfrak{D}_n}$.

THEOREM

Let T be a finitary secured tree in $L_{\omega\omega}$. There is then an n with $\mathcal{H}_n(T)$ secured.

Proof: We know that there exists a finite $\mathfrak{D}$ with $\mathcal{H}_{\mathfrak{D}}(T)$ secured. But each finite $\mathfrak{D}$ is contained in an $\mathfrak{D}_n$, and the result follows. Ω

COROLLARY

For $\Gamma \to \Delta$ in $L_{\omega\omega}$:

$$\vdash \Gamma \to \Delta \iff \exists n \vdash H_n(\Gamma \to \Delta)$$

This is a usual formulation of the Herbrand theorem in $L_{\omega\omega}$.

<u>6.6.2.</u> The language $L_{\kappa\omega}$ $\kappa > \omega$ are easier to treat. Let $\mathfrak{D}$ be a sequence without repetition of all terms in the language. Then $\mathfrak{D}$ is obviously a Herbrand domain. Put $\mathcal{H} = \mathcal{H}_{\mathfrak{D}}$ $H = H_{\mathfrak{D}}$.

THEOREM

$\mathcal{H}$ is a classical isomorphism in $L_{\kappa\omega}$, $\kappa > \omega$.

Proof: We observe that in constructing $\mathcal{H}(T)$ from $\mathcal{S}(T)$ we do not change any atomic formulas. So secured nodes in $\mathcal{S}(T)$ go over into secured nodes in $\mathcal{H}(T)$. Hence $\mathcal{H}$ is a derivability morphism. We know already that $\mathcal{H}$ is a falsifiability morphism. Ω

<u>6.6.3.</u> Now we come to the admissible languages $L_A \neq L_{\omega\omega}$. Here we let $\mathfrak{D}^*$ be an A-recursive sequence without repetition of all terms in the language. Then $\mathfrak{D}^*$ is a Herbrand domain. Put $\mathcal{H} = \mathcal{H}_{\mathfrak{D}^*}$ $H = H_{\mathfrak{D}^*}$

THEOREM

$\mathcal{H}$ is a classical isomorphism in L_A, $L_A \neq L_{\omega\omega}$.

The proof is as above.

Since the finitary analyzing theorem is true for $L_A \subseteq L_{\omega_1\omega}$ we can conclude :

COROLLARY

For any A-finite sequent $\Gamma \to \Delta$ $(L_{\omega\omega} \subsetneq L_A \subseteq L_{\omega_1\omega})$. There is an A-finite derivation of $\Gamma \to \Delta$ if and only if there is an A-finite derivation of $H(\Gamma \to \Delta)$.

<u>6.7.</u> As a sidestep in the development here we note that the construction of $\mathcal{H}_{\mathcal{D}}(T)$ from $\mathcal{S}(T)$ can be inverted.

THEOREM

There is a derivability morphism $\mathcal{M}_{\mathcal{D}}$ defined on classical trees over $\mathcal{D}$-Herbrand transforms such that

i) If T is a classical tree over $H_{\mathcal{D}}(\Gamma \to \Delta)$, then $\mathcal{M}_{\mathcal{D}}(T)$ is a classical tree over $\mathcal{S}(T)$.

ii) $\mathcal{M}_{\mathcal{D}}$ is a derivability morphism.

iii) $\mathcal{H}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}(T) = T$

The proof is left as an exercise.

I do not know whether a morphism with similar properties can be defined between classical trees over $S(\Gamma \to \Delta)$ and classical trees over $\Gamma \to \Delta$. (See 8.3).

7. EQUALITY.

7.1. So far we have not treated $L_{\kappa\omega}$ with equality. This can easily be done in our setting. First we must define the logic $L^{=}_{\kappa\omega}$.

LANGUAGE

We add one new binary relationsymbol = . Else the language of $L^{=}_{\kappa\omega}$ is like $L_{\kappa\omega}$.

RULES

The rules in the derivations of $L^{=}_{\kappa\omega}$ are exactly the same as those of $L_{\kappa\omega}$.

AXIOMS

The axioms are all sequents which can be got from sequents $\Gamma \to F$ by application of a structural rule where

i) The formulas in Γ and the formula F are atomic; and

ii) $\Gamma \to F$ is valid in ordinary first order logic with equality ($L^{=}_{\omega\omega}$).

A syntachtical description of the sequents $\Gamma \to F$ above is given in FEFERMAN 1968. For our purposes here we only need to note that the three properties of axioms in 5.7 are satisfied. Property 3 is a consequence of the compactness theorem for $L^{=}_{\omega\omega}$.

In the Skolem theory and the Herbrand theory above for $L_{\kappa\omega}$ we have only used the three properties for axioms. Hence we can without any changes carry over the whole theory to $L^{=}_{\kappa\omega}$.

GENERAL RESULT

The theorems above in chapter 5 and 6, which hold for $L_{\kappa\omega}$, hold for $L^{=}_{\kappa\omega}$.

7.2. An interesting sidepath opens now. We can give other axiomatic systems with axioms satisfying properties 1-3. An example is the theory of fields with $0,1,+,-,\times,^{-1}$. We shall not pursue this here.

8. CONCLUSION.

8.1. Comparison with HERBRAND 1929.

I shall first explain what Herbrand did. First we put as usual $\exists x$ for $\neg\forall x\neg$. To each formula F in $L_{\omega\omega}$ we get a prenex formula F^* by applying transformations like the below on formulaparts :

$$\neg\forall xGx \to \exists x\neg Gx \;; \qquad \forall xGx \to \forall yGy, \quad y \text{ not in } Gx.$$

$$\neg\exists xGx \to \forall x\neg Gx \;; \qquad \exists xGx \to \exists yGy, \quad y \text{ not in } Gx$$

$$\forall xGx \wedge H \to \forall x(Gx \wedge H)$$

$$\exists xGx \wedge H \to \exists x(Gx \wedge H)\;, \quad x \text{ not in } H$$

Conversely we get the canonical form F^+ by applying the inverse transformations. In the canonical form the quantifiers have as small range as possible.

Now to each formula F, Herbrand gives the following three properties :

A: There is a derivation of F^* which can be separated into two parts - first a proportional part and then a quantificational part.

B: There is an n and a derivation of $H_n(\rightarrow F)$.

C: There is an n and a derivation of $H_n(\rightarrow F^+)$.

Herbrand has a faulty proof (see DREBEN, ANDREWS, AND AANDERAA 1963) that each of these properties are equivalent to $\vdash \rightarrow F$. Since the proof can be repaired (DREBEN AND DENTON 1966) we disregard the errors in this discussion.

Property A reminds one of Gentzens verschäfter Hauptsatz (GENTZEN 1934). Gentzen claims correctly in a note in his paper that his result is a generalization of Herbrands result for property A, since it gives property A not only for prenex formulas, but for sequents of prenex formulas. On the other hand it is not quite correct to say that Gentzens result gives the Herbrand theorem (property B), especially so since Herbrand distinguishes between property B and property C and thereby stresses that we work with any formula and not only prenex formulas (or sequents of prenex formulas).

The above has been stressed by van Heijenoort in his introductions to Herbrands collected logical works. (HERBRAND 1968) and to the English translation of the last chapter of HERBRAND 1929 in VAN HEIJENOORT 1967. There are certainly more to Herbrands work than just this difference. It is not hard to see that if we can prove the Herbrand theorem (property B) for all prenex formulas then we can prove it for all formulas. The argument below gives an indication of how this can be done.

1. Assume the Herbrand theorem for all prenex formulas.

2. We then get the Skolem theorem for all prenex formulas.

3. We now take as example the formula $\exists y \forall x R(x,y) \wedge \exists z \forall u S(z,u)$

Then $\vdash \exists y \forall x R(x,y) \wedge \exists z \forall u S(z,u)$

$\Leftrightarrow \ \vdash \exists y \forall x \exists z \forall u [R(x,y) \wedge S(z,u)]$

$\Leftrightarrow \ \vdash \exists y \exists z [R(f(y),y) \wedge S(z,g(y,z))]$

$\Leftrightarrow \ \vdash \exists y \exists z \forall u [R(f(y),y) \wedge S(z,u)]$

$\Leftrightarrow \ \vdash \exists y R(f(y),y) \wedge \exists z \forall u S(z,u)$

$\Leftrightarrow \ \vdash \exists z \forall u \exists y [R(f(y),y) \wedge S(z,u)]$

$\Leftrightarrow \ \vdash \exists z \exists y [R(f(y),y) \wedge S(z,h(z))]$

$\Leftrightarrow$ Exists $N \vdash \bigvee_{z \in \mathfrak{D}_N} \bigvee_{y \in \mathfrak{D}_N} |R(f(y),y) \wedge S(z,h(z))|$

$\Leftrightarrow$ Exists $N \vdash \bigvee_{y \in \mathfrak{D}_N} R(f(y),y) \wedge \bigvee_{z \in \mathfrak{D}_N} S(z,h(z))$

In the above $\mathfrak{D}_N$ is a finite sequence of terms of length $\leq N$ built up from e,f,h.

So we have the Herbrand theorem for the formula.

We conclude that the importance of Herbrand work does not lie in the handling of non-prenex formulas.

Herbrand himself stressed that this theorem should be regarded as a metamathematical version of Löwenheims theorem. ("Metamathematical" here means "finitary".) This comes out clearly in the version of Herbrand theorem in DREBEN AND DENTON 1970 :

(a) There is a uniform way to find (primitive recursively) a tantalogous validity expansion for any logical theorem A from any logical proof of A.

(b) There is a uniform way to find (primitive recursively) a logical proof for a formula A from any tantalogous validity expansion of A.

Instead of "tantalogous validity expansion of A" we can write "an N and a proof of $H_N(\to A)$".

Neither Herbrand nor Dreben and Denton have anything more to say about "the uniform way to find (primitive recursively)". So their explanation is unsatisfying as it stands.

I think that all the uniformity we want in the Skolem and Herbrand theorems is expressed with the classical morphisms $\mathcal{S}$ and $\mathcal{H}_n$. If we want to get from $\vdash H_N(A)$ to $\vdash A$, then this can be done by using the analyzing theorem for the sequent $\to A$. I find it hard to get more uniformity between the derivation of $H_N(A)$ and the derivation of A than can be got by this kind of general argument. A better argument should give a derivability morphism between finitary derivation of $\mathcal{H}_{\mathcal{D}}(A)$ in $L_{\kappa\omega}$, $\kappa > \omega_1$ and finitary derivation of A. (This problem is still open).

8.2. Comparison with NEBRES 1970.

Nebres discusses the Herbrand theorem. He distinguishes between semantic and syntactic versions of it in $L_{\omega\omega}$. The semantic versions are (following Nebres) :

(a) For every formula F, there is an existential formula F^* such that F is valid iff F^* is valid.

(b) A prenex existential formula $\exists x_1 \dots \exists x_n F(x_1, \dots, x_n)$ is valid iff it is valid in all canonical (term) models.

(b') A prenex existential formula $\exists x_1 \dots \exists x_n F(x_1, \dots, x_n)$ is valid iff for some finite set T of terms $\bigvee_{t_1, \dots, t_n \in T} F(t_1, \dots, t_n)$ is valid.

The proof of (a) is the usual proof of the Skolem theorem. (We let F^* be S(F). It obviously generalizes to any classical logic. (b) is proved by some easy lemmas. It follows also by the fact that each model has a canonical (term) model as submodel. (b') follows from (b) by an application of compactness. In the generalizations to other logics we talk about "existential formula" instead of "prenex existential formula". The results for $L_{\omega\omega}$ hold with this change without altering the argument. The generalizations to the admissible languages L_A, $L_{\omega\omega} \subsetneq L_A \subseteq L_{\omega_1\omega}$, are clear if we relativize finite to the logic. It is particularly simple in those logics since the canonical (term) model built up from A-finitely many symbols is A-finite.

The syntactic versions are (following Nebres) :

(a) There are functionals (in an explicitly defined class) which map finitary derivations of formula F into finitary derivations of some existential formula F^*, and finitary derivation of prenex existential formula $x_1 \ldots x_n G(x_1, \ldots, x_n)$ into finite sets T of terms and finitary derivations of

$$\bigvee_{t_1, \ldots, t_n \in T} G(t_1, \ldots, t_n)$$

(b) for any finitary derivation $\mathcal{D}$ of a prenex formula, there is another finitary derivation $\mathcal{D}'$ of the formula in which all propositional inferences are above quantifierinferences.

Nebres proves the generalization of (a) to countable languages, and shows that (b) fails in countable languages different from $L_{\omega\omega}$, (even if we delete the last finitary). How does this compare with my work ? I make the following remarks :

1. The finitary derivations mentioned above may contain cuts. This does not fit into my framework. The argument for the functional in the first part of (a) is not affected by this. For the last part one needs an extra argument á la the one in Nebres. (The argument is not hard.)
   For the remaining remarks we restrict ourselves to cut-free derivations.

2. Not all cut-free derivations are secured classical trees. (See section 4.2.) Our theory is made for (secured) classical trees, while Nebres seems to work with cut-free derivations. There is a mistake in his proof of the generalization of the first part of (a). The natural thing to assume to make it work is that we have secured classical trees and not derivations.

3. Nebres states that his functionals are in an explicitly defined class - but he does not explicitly give this class. This is done in my work with $\mathcal{S}$ and $\mathcal{H}_{\mathcal{D}}$. Using them we can state (a) as :

   i) $\mathcal{S}$ is a derivability morphism.

   ii) For each finitary secured T, there is an Herbrand domain $\mathcal{D}$ with $\mathcal{H}_{\mathcal{D}}(T)$ secured.

4. With the reformulation of (a) in 3 it is clear that Nebres has in his syntactic version only given part of the Herbrand theorem. In fact he has only given the easy part - the part that is true in both intuitionistic logic and modal logics (M or S4).

8.3. Open problems.

PROBLEM 1

Define a derivability morphism $\mathcal{M}$, which to the classical tree T over $S(\Gamma \to \Delta)$ gives the classical tree $\mathcal{M}(T)$ over $\Gamma \to \Delta$.

PROBLEM 2

Define if possible a derivability morphism as in problem 1 which transforms finitary trees over into finitary trees.

PROBLEM 3

Is it true that for any finite sequent $\Gamma \to \Delta$ in $L_{\kappa\omega}$ that $\Gamma \to \Delta$ that $\Gamma \to \Delta$ is finitary derivable if $S(\Gamma \to \Delta)$ is finitary derivable?

As problem 1 and problem 2 stand, both are trivially impossible. This is because a derivation of $S(\Gamma \to \Delta)$ may be shorter than any derivation of $\Gamma \to \Delta$. The problems become non-trivial if we relax the definition of morphism by not requiring the tree-structure to be preserved. The open problems are the problems with the weaker definition of morphism.

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